

Modal and temporal DPLs

Konstantine Arkoudas

March 23, 2001

Abstract

This paper formulates a variety of propositional modal logics as DPLs [1]. In particular, we introduce DPLs \mathcal{K} , \mathcal{T} , \mathcal{D} , \mathcal{S}_4 , and \mathcal{S}_5 , modelling the well-known logics K, T, D, S4, and S5 [5, 3, 2, 4]. The formal syntax and semantics of each DPL is specified in detail, sample deductions are given, the associated proof theory is discussed, and the soundness and completeness of each system is formally proved.

1.1 \mathcal{K}

Propositions have the same abstract syntax as in zero-order $\mathcal{CN}\mathcal{D}$, with the addition of two new unary propositional constructors, the modal operators \Box and \Diamond :

$$P ::= A \mid \mathbf{true} \mid \mathbf{false} \mid \neg P \mid P \wedge Q \mid P \vee Q \mid P \Rightarrow Q \mid P \Leftrightarrow Q \mid \Box P \mid \Diamond P \quad (1.1)$$

where A again ranges over a countable infinity of unspecified atoms. As before, we will use the letters A , B , and C as typical atoms, and P , Q , and R as propositions. We will observe the same precedence and associativity conventions that we laid down in Section ??, with the added stipulation that \Box and \Diamond have higher precedence (bind tighter) than all of the other logical connectives, so that, for instance, $\Box P \wedge Q$ stands for $(\Box P) \wedge Q$ rather than $\Box(P \wedge Q)$. Parentheses and brackets will often be used to enhance readability. A convenient notation for expressing propositions with “iterated modalities” (nested applications of modal operators) is $\Box^i P$, where $\Box^0 P$ is defined as P and $\Box^{i+1} P$ as $\Box \Box^i P$ (and likewise for $\Diamond^i P$).

The deductions of \mathcal{K} have the same the structure as before, save for the presence of one additional syntax form: **nec**: D . This form will capture the so-called “necessitation rule” of modal logic, by which one may derive $\Box P$ whenever P has been shown to be a theorem:

$$D ::= P \mid \mathit{Prim-Rule} P_1, \dots, P_n \mid \mathbf{assume} P \mathbf{in} D \mid D_1; D_2 \mid \mathbf{nec}: D$$

We will refer to deductions of the form **nec**: D as *necessitations*. The definitions of terms such as “primitive deduction” and notions such as deduction size, conclusion-annotated form, etc., carry over from $\mathcal{CN}\mathcal{D}$ [1] either directly or with obvious modifications.

The collection of primitive inference rules over which *Prim-Rule* ranges in the foregoing grammar is as specified in Figure ??, with the addition of three new primitive inference rules: \Diamond -**intro**, \Diamond -**elim**, and **K**.¹ As their names imply, the first two rules are

¹The name **K** was first used by Lemmon and Scott [6] in honor of Saul Kripke, the founder of formal possible-world semantics.

just introduction and elimination rules for the possibility operator.² Their respective semantics are given by the following two evaluation axioms:

$$\beta \cup \{\neg\Box\neg P\} \vdash \diamond\text{-intro} \neg\Box\neg P \rightsquigarrow \diamond P \quad (1.2)$$

$$\beta \cup \{\diamond P\} \vdash \diamond\text{-elim} \diamond P \rightsquigarrow \neg\Box\neg P \quad (1.3)$$

The semantics of **K** are specified by the following evaluation axiom:

$$\beta \cup \{\Box(P \Rightarrow Q)\} \vdash \mathbf{K} \Box(P \Rightarrow Q) \rightsquigarrow \Box P \Rightarrow \Box Q. \quad (1.4)$$

This simply says that if we know P to necessarily imply Q , then if P is necessary then so is Q . This is often formulated as a single axiom, given by the formula

$$\Box(P \Rightarrow Q) \Rightarrow (\Box P \Rightarrow \Box Q) \quad (1.5)$$

but our casting of it as a unary inference rule is more in the DPL spirit and is easily seen to subsume the axiomatic formulation, as 1.5 is readily derivable thus:

assume $\Box(P \Rightarrow Q)$ **in**

K $\Box(P \Rightarrow Q)$

The rest of the primitive rules have the same semantics as before (specified in Figure ??).

The evaluation semantics of non-primitive deductions are also adapted from \mathcal{CND} without change (Figure ??). We only need to add a new rule giving the semantics of necessitations:

$$\frac{\emptyset \vdash D \rightsquigarrow P}{\beta \vdash \mathbf{nec}: D \rightsquigarrow \Box P} \quad [R_6]$$

This rule says that if a conclusion P is derived from no assumptions whatsoever (i.e., from the empty assumption base), then P is necessary. Of course in the case of \mathcal{CND} we know that if P is derived from the empty assumption base then it is a tautology, i.e., logically valid, and in that light $[R_6]$ reflects the intuition that tautologies are necessary truths. But note that the body of a necessitation can itself be a necessitation, so $[R_6]$ entails not only that all tautologies of propositional logic are necessary, but also that they are necessarily necessary. In fact once we have proven a tautology P then we can easily derive $\Box^n P$ for any $n \geq 0$ by iterating the necessitation rule sufficiently

²It is more customary to “define” the possibility operator in terms of \Box by introducing $\diamond P$ as an abbreviation for $\neg\Box\neg P$, but then an additional rule must be adopted that allows us to apply the definition to arbitrary parts of a given proposition. Introduction and elimination rules are better suited to DPLs.

many times. We stress that this holds only for tautologies, not for arbitrary P —we cannot, in general, prove $\Box P$ once we have proven P .

It is for this reason that our necessitation rule $[R_6]$ is called “weak necessitation”. There are alternative formulations of modal logic with “strong” necessitation rules, whereby one is entitled to infer $\Box P$ once P is derived from *any* set of assumptions, not just the empty set. In our setting such a stronger necessitation rule could be formulated as follows:³

$$\frac{\beta \vdash D \rightsquigarrow P}{\beta \vdash \mathbf{nec}: D \rightsquigarrow \Box P} \quad [R'_6]$$

But this would mean that $P \Rightarrow \Box P$ is derivable for any P , via the simple deduction

assume P in
nec: P

Clearly, this is unacceptable under any intuitive modal interpretation of \Box . If we construe \Box as “necessarily”, then $P \Rightarrow \Box P$ certainly should not hold for contingent propositions: the mere fact that P happens to be true does not make it necessary. Otherwise we would have to conclude that everything that is true is necessarily true. Under a temporal interpretation of \Box , $P \Rightarrow \Box P$ would mean that if P is true at some given instant then it will always be true, which clearly ought to be false. The formula $P \Rightarrow \Box P$ is likewise untenable under all other well-known interpretations of the modal operators (e.g., epistemic interpretations). Indeed, once we introduce formal Kripke semantics we will be able to show that this formula is not valid.⁴ Thus a rule such as $[R'_6]$ would render our language unsound.

In systems that do have strong necessitation rules this unwanted conclusion is averted by the failure of the “deduction theorem”, which says that if Q can be derived from $\Phi \cup \{P\}$ then $P \Rightarrow Q$ can be derived from Φ . For if the deduction theorem did hold in the presence of strong necessitation then we could easily deduce $P \Rightarrow \Box P$ for any P : first derive P from $\{P\}$, then apply strong necessitation to obtain $\Box P$ from $\{P\}$, and then infer $P \Rightarrow \Box P$ from \emptyset by virtue of the deduction theorem. By contrast, in systems with weak necessitation rules the deduction theorem usually continues to hold. In any case, even though the derivability relation $\Phi \vdash P$ is not the same for weak and strong systems (e.g. $\{P\} \vdash \Box P$ holds in strong systems but not in weak ones), the two coincide when $\Phi = \emptyset$. As a result, both kinds of systems turn out to be complete and hence equally powerful because, as we will see shortly, completeness is

³In fact strong necessitation could be captured by a simple primitive inference rule, say **nec**, with the following semantics: $\beta \cup \{P\} \vdash \mathbf{nec} P \rightsquigarrow \Box P$. We need not introduce a special syntax form **nec: D** for it.

⁴For \mathcal{K} , this means that there are frames in which $P \Rightarrow \Box P$ is not valid.

formulated in terms of logical validity—entailment from no assumptions—rather than general logical consequence.

Finally, let us fix some notation that we will use throughout the rest of this paper. We will write $\vdash_{\mathcal{K}} \beta \vdash D \rightsquigarrow P$ (or just $\beta \vdash D \rightsquigarrow P$ if there is no risk of confusion) to mean that the judgement $\beta \vdash D \rightsquigarrow P$ is derivable from the primitive-rule axioms of Figure ?? and 1.4 along with rules $[R_1]$ — $[R_6]$. That is, $\vdash_{\mathcal{K}} \beta \vdash D \rightsquigarrow P$ iff there is a sequence

$$\beta_1 \vdash D_1 \rightsquigarrow P_1, \dots, \beta_n \vdash D_n \rightsquigarrow P_n$$

such that $\beta_n = \beta, D_n = D, P_n = P$, and for each $i = 1, \dots, n$, the judgment $\beta_i \vdash D_i \rightsquigarrow P_i$ is either an instance of one of the axioms $[R_1]$, $[R_2]$, or $[R_3]$; or an instance of one of the axioms in Figure ?? or 1.4; or else follows from previous judgments through $[R_4]$, or $[R_5]$, or $[R_6]$. For any set of propositions Φ , we write $\Phi \vdash_{\mathcal{K}} P$ iff there is a deduction D and an assumption base $\beta \subseteq \Phi$ such that $\vdash_{\mathcal{K}} \beta \vdash D \rightsquigarrow P$. For $\Phi = \emptyset$ we write $\vdash_{\mathcal{K}} P$ as an abbreviation for $\emptyset \vdash_{\mathcal{K}} P$.

1.1.1 Examples

$$\boxed{\Box(P \wedge Q) \Rightarrow \Box P \wedge \Box Q}$$

Proof:

```

assume  $\Box(P \wedge Q)$  in
  begin
    nec: assume  $P \wedge Q$  in
      left-and  $P \wedge Q$ ;
      K  $\Box(P \wedge Q \Rightarrow P)$ ;
      modus-ponens  $\Box(P \wedge Q) \Rightarrow \Box P, \Box(P \wedge Q)$ ;
      nec: assume  $P \wedge Q$  in
        right-and  $P \wedge Q$ ;
        K  $\Box(P \wedge Q \Rightarrow Q)$ ;
        modus-ponens  $\Box(P \wedge Q) \Rightarrow \Box Q, \Box(P \wedge Q)$ ;
        both  $\Box P, \Box Q$ 
      end
    end
  
```

The proof is perhaps easier to follow in conclusion-annotated form:

Proof:

```

assume  $\Box(P \wedge Q)$  in
  begin

```

$\Box(P \wedge Q \Rightarrow P)$ by
 nec: assume $P \wedge Q$ **in**
 left-and $P \wedge Q$;
 $\Box(P \wedge Q) \Rightarrow \Box P$ by **K** $\Box(P \wedge Q \Rightarrow P)$;
 $\Box P$ by **modus-ponens** $\Box(P \wedge Q) \Rightarrow \Box P, \Box(P \wedge Q)$;
 $\Box(P \wedge Q \Rightarrow Q)$ by
 nec: assume $P \wedge Q$ **in**
 right-and $P \wedge Q$;
 $\Box(P \wedge Q) \Rightarrow \Box Q$ by **K** $\Box(P \wedge Q \Rightarrow Q)$;
 $\Box Q$ by **modus-ponens** $\Box(P \wedge Q) \Rightarrow \Box Q, \Box(P \wedge Q)$;
 $\Box P \wedge \Box Q$ by **both** $\Box P, \Box Q$
end

$$\boxed{(\Box P \wedge \Box Q) \Rightarrow \Box(P \wedge Q)}$$

Proof:

assume $\Box P \wedge \Box Q$ **in**
 begin
 nec: assume P **in**
 assume Q **in**
 both P, Q ;
 K $\Box(P \Rightarrow Q \Rightarrow P \wedge Q)$;
 left-and $\Box P \wedge \Box Q$;
 modus-ponens $\Box P \Rightarrow \Box(Q \Rightarrow P \wedge Q), \Box P$;
 K $\Box(Q \Rightarrow P \wedge Q)$;
 right-and $\Box P \wedge \Box Q$;
 modus-ponens $\Box Q \Rightarrow \Box(P \wedge Q), \Box Q$
 end

And in conclusion-annotated form:

Proof:

assume $\Box P \wedge \Box Q$ **in**
 begin
 $\Box(P \Rightarrow Q \Rightarrow P \wedge Q)$ by
 nec: assume P **in**
 assume Q **in**
 both P, Q ;
 end

$\Box P \Rightarrow \Box(Q \Rightarrow P \wedge Q)$ by **K** $\Box(P \Rightarrow Q \Rightarrow P \wedge Q)$;
 $\Box P$ by **left-and** $\Box P \wedge \Box Q$;
 $\Box(Q \Rightarrow P \wedge Q)$ by **modus-ponens** $\Box P \Rightarrow \Box(Q \Rightarrow P \wedge Q), \Box P$;
 $\Box Q \Rightarrow \Box(P \wedge Q)$ by **K** $\Box(Q \Rightarrow P \wedge Q)$;
 $\Box Q$ by **right-and** $\Box P \wedge \Box Q$;
 $\Box(P \wedge Q)$ by **modus-ponens** $\Box Q \Rightarrow \Box(P \wedge Q), \Box Q$
end

We will present the remaining examples in direct style only, leaving their formulation in conclusion-annotated form to the interested reader.

Lemma 1.1 *If $\vdash_{\mathcal{K}} P \Leftrightarrow Q$ then $\vdash_{\mathcal{K}} \neg P \Leftrightarrow \neg Q$. Further, $\vdash_{\mathcal{K}} (P_1 \circ P_2) \Leftrightarrow (P'_1 \circ P'_2)$ whenever $\vdash_{\mathcal{K}} P_1 \Leftrightarrow P'_1$ and $\vdash_{\mathcal{K}} P_2 \Leftrightarrow P'_2$, for $\circ \in \{\wedge, \vee, \Rightarrow, \Leftrightarrow\}$.*

Proof: A straightforward exercise in propositional reasoning. We illustrate with conjunctions: assuming $\vdash_{\mathcal{K}} P_1 \Leftrightarrow P'_1$ and $\vdash_{\mathcal{K}} P_2 \Leftrightarrow P'_2$, so that there are deductions D_1 and D_2 such that $\emptyset \vdash_{\mathcal{K}} D_1 \rightsquigarrow P_1 \Leftrightarrow P'_1$ and $\emptyset \vdash_{\mathcal{K}} D_2 \rightsquigarrow P_2 \Leftrightarrow P'_2$, the following deduction derives $P_1 \wedge P_2 \Leftrightarrow P'_1 \wedge P'_2$:

D_1 ;
left-iff $P_1 \Leftrightarrow P'_1$;
right-iff $P_1 \Leftrightarrow P'_1$;
 D_2 ;
left-iff $P_2 \Leftrightarrow P'_2$;
right-iff $P_2 \Leftrightarrow P'_2$;
assume $P_1 \wedge P_2$ **in**
 begin
 left-and $P_1 \wedge P_2$;
 modus-ponens $P_1 \Rightarrow P'_1, P_1$;
 right-and $P_1 \wedge P_2$;
 modus-ponens $P_2 \Rightarrow P'_2, P_2$;
 both P'_1, P'_2
 end;
assume $P'_1 \wedge P'_2$ **in**
 begin
 left-and $P'_1 \wedge P'_2$;
 modus-ponens $P'_1 \Rightarrow P_1, P'_1$;
 right-and $P'_1 \wedge P'_2$;
 modus-ponens $P'_2 \Rightarrow P_2, P'_2$;
 end

both P_1, P_2
end;
equivalence $P_1 \wedge P_2 \Rightarrow P'_1 \wedge P'_2, P'_1 \wedge P'_2 \Rightarrow P_1 \wedge P_2$

The rest of the cases are handled similarly. ■

Lemma 1.2 *If $\vdash_{\mathcal{K}} P \Leftrightarrow Q$ then $\vdash_{\mathcal{K}} \Box P \Leftrightarrow \Box Q$.*

Proof: Assuming $\vdash_{\mathcal{K}} P \Leftrightarrow Q$, so that there is a D such that $\emptyset \vdash D \rightsquigarrow P \Leftrightarrow Q$, the following deduction derives $\Box P \Leftrightarrow \Box Q$ from the empty assumption base:

nec: D ;
nec: assume $P \Leftrightarrow Q$ **in**
 left-iff $P \Leftrightarrow Q$;
K $\Box[(P \Leftrightarrow Q) \Rightarrow (P \Rightarrow Q)]$;
modus-ponens $\Box(P \Leftrightarrow Q) \Rightarrow \Box(P \Rightarrow Q), \Box(P \Leftrightarrow Q)$;
K $\Box(P \Rightarrow Q)$;
nec: assume $P \Leftrightarrow Q$ **in**
 right-iff $P \Leftrightarrow Q$;
K $\Box[(P \Leftrightarrow Q) \Rightarrow (Q \Rightarrow P)]$;
modus-ponens $\Box(P \Leftrightarrow Q) \Rightarrow \Box(Q \Rightarrow P), \Box(P \Leftrightarrow Q)$;
K $\Box(Q \Rightarrow P)$;
equivalence $\Box P \Rightarrow \Box Q, \Box Q \Rightarrow \Box P$

■

Lemma 1.3 *If $\vdash_{\mathcal{K}} P \Leftrightarrow Q$ then $\vdash_{\mathcal{K}} \Diamond P \Leftrightarrow \Diamond Q$.*

Proof: By two applications of Lemma 1.1 and one of Lemma 1.2, the assumption $\vdash_{\mathcal{K}} P \Leftrightarrow Q$ yields

$$\vdash_{\mathcal{K}} \neg \Box \neg P \Leftrightarrow \neg \Box \neg Q.$$

Therefore, there is a D such that

$$\emptyset \vdash D \rightsquigarrow \neg \Box \neg P \Leftrightarrow \neg \Box \neg Q.$$

Now let D' be the following:

D ;
left-iff $\neg \Box \neg P \Leftrightarrow \neg \Box \neg Q$;
right-iff $\neg \Box \neg P \Leftrightarrow \neg \Box \neg Q$;

```

assume  $\diamond P$  in
  begin
     $\diamond$ -elim  $\diamond P$ ;
    modus-ponens  $\neg \Box \neg P \Rightarrow \neg \Box \neg Q, \neg \Box \neg P$ ;
     $\diamond$ -intro  $\neg \Box \neg Q$ 
  end;
assume  $\diamond Q$  in
  begin
     $\diamond$ -elim  $\diamond Q$ ;
    modus-ponens  $\neg \Box \neg Q \Rightarrow \neg \Box \neg P, \neg \Box \neg Q$ ;
     $\diamond$ -intro  $\neg \Box \neg P$ 
  end;
equivalence  $\diamond P \Rightarrow \diamond Q, \diamond Q \Rightarrow \diamond P$ 

```

It is readily verified that $\emptyset \vdash D' \rightsquigarrow \diamond P \Leftrightarrow \diamond Q$. ■

Next, let g be any (computable) function from propositions to propositions. We define a functional \mathcal{F} that applies such a g to a proposition P in a structure-preserving way:

$$\begin{aligned}
\mathcal{F}(A, g) &= g(A) \\
\mathcal{F}(\neg P, g) &= \neg g(P) \\
\mathcal{F}(\Box P, g) &= \Box g(P) \\
\mathcal{F}(\diamond P, g) &= \diamond g(P) \\
\mathcal{F}(P_1 \circ P_2, g) &= g(P_1) \circ g(P_2)
\end{aligned}$$

for $\circ \in \{\wedge, \vee, \Rightarrow, \Leftrightarrow\}$. We now define $P[P_2/P_1]$ as the proposition obtained from P by replacing every occurrence of P_1 by P_2 . Formally:

$$P[P_2/P_1] = P = P_1? \rightarrow P_2, \text{atom?}(P) \rightarrow P, \mathcal{F}(P, \lambda P.P[P_2/P_1])$$

where atom? returns true or false depending on whether the given proposition is an atom.

Lemma 1.4 *If $\vdash_{\mathcal{K}} P_1 \Leftrightarrow P_2$ then $\vdash_{\mathcal{K}} P \Leftrightarrow P[P_2/P_1]$.*

Proof: By induction on the structure of P . When P is an atom the result is immediate. When P is a negation $\neg Q$, either $P_1 = P$ or not. If $P_1 = P$ we have to prove $\vdash_{\mathcal{K}} P_1 \Leftrightarrow P_2$, but that holds by supposition. If $P_1 \neq P$ then we have to show that

$$\vdash_{\mathcal{K}} \neg Q \Leftrightarrow \neg(Q[P_2/P_1]). \quad (1.6)$$

The inductive hypothesis gives $\vdash_{\mathcal{K}} Q \Leftrightarrow Q[P_2/P_1]$, and now 1.6 follows from Lemma 1.1. The remaining propositional cases are treated similarly. When P is of the form $\Box Q$, again either $P_1 = P$ or not. If $P_1 = P$ the result holds by supposition, so suppose $P_1 \neq P$. In that case we need to show

$$\vdash_{\mathcal{K}} \Box Q \Leftrightarrow \Box(Q[P_2/P_1]). \quad (1.7)$$

The inductive hypothesis gives $\vdash_{\mathcal{K}} Q \Leftrightarrow Q[P_2/P_1]$, and now 1.7 follows from Lemma 1.2. Finally, when P is of the form $\Diamond Q$ the reasoning is similar, but we use Lemma 1.3 in place of Lemma 1.2. \blacksquare

Lemma 1.5 *For all P :*

- (a) $\vdash_{\mathcal{K}} \neg\Box\neg P \Leftrightarrow \Diamond P$; (b) $\vdash_{\mathcal{K}} \Box P \Leftrightarrow \neg\Diamond\neg P$;
(c) $\vdash_{\mathcal{K}} \neg\Box P \Leftrightarrow \Diamond\neg P$; (d) $\vdash_{\mathcal{K}} \neg\Diamond P \Leftrightarrow \Box\neg P$.

Proof: Part (a) is immediate by \Diamond -**intro** and \Diamond -**elim**. For (b), we clearly have $\vdash_{\mathcal{K}} P \Leftrightarrow \neg\neg P$, so Lemma 1.4 implies $\vdash_{\mathcal{K}} \Box P \Leftrightarrow \Box\neg\neg P$. Accordingly, let D be a deduction such that $\emptyset \vdash D \rightsquigarrow \Box P \Leftrightarrow \Box\neg\neg P$. Then, in one direction, let D_1 be the following:

D ;
left-iff $\Box P \Leftrightarrow \Box\neg\neg P$;
assume $\Box P$ **in**
 begin
 modus-ponens $\Box P \Rightarrow \Box\neg\neg P, \Box P$;
 suppose-absurd $\Diamond\neg P$ **in**
 begin
 \Diamond -**elim** $\Diamond\neg P$;
 absurd $\Box\neg\neg P, \neg\Box\neg\neg P$
 end
 end
 end

It is readily verified that $\emptyset \vdash D_1 \rightsquigarrow \Box P \Rightarrow \neg\Diamond\neg P$. In the converse direction, the fact that $\vdash_{\mathcal{K}} P \Leftrightarrow \neg\neg P$ entails (again via Lemma 1.4) that $\vdash_{\mathcal{K}} \neg\Box P \Leftrightarrow \neg\Box\neg\neg P$. Therefore, there is a deduction D' such that

$$\emptyset \vdash D' \rightsquigarrow \neg\Box P \Leftrightarrow \neg\Box\neg\neg P. \quad (1.8)$$

Now let D_2 be the following:

D' ;
left-iff $\neg\Box P \Leftrightarrow \neg\Box\neg\neg P$;
assume $\neg\Diamond\neg P$ **in**
 begin
 suppose-absurd $\neg\Box P$
 begin
 modus-ponens $\neg\Box P \Rightarrow \neg\Box\neg\neg P, \neg\Box P$;
 \Diamond -intro $\neg\Box\neg\neg P$;
 absurd $\Diamond\neg P, \neg\Diamond\neg P$
 end;
 double-negation $\neg\neg\Box P$
 end

The reader will verify that $\emptyset \vdash D_2 \rightsquigarrow \neg\Diamond\neg P \Rightarrow \Box P$, and therefore that

$$\emptyset \vdash D_1; D_2; \mathbf{equivalence} \Box P \Rightarrow \neg\Diamond\neg P, \neg\Diamond\neg P \Rightarrow \Box P \rightsquigarrow \Box P \Leftrightarrow \neg\Diamond\neg P.$$

This establishes (b).

For one direction of (c): as before, let D' be the deduction described by 1.8, and let D_1 be the following:

D' ;
left-iff $\neg\Box P \Leftrightarrow \neg\Box\neg\neg P$;
assume $\neg\Box P$ **in**
 begin
 modus-ponens $\neg\Box P \Rightarrow \neg\Box\neg\neg P, \neg\Box P$;
 \Diamond -intro $\neg\Box\neg\neg P$
 end

Clearly, $\emptyset \vdash D_1 \rightsquigarrow \neg\Box P \Rightarrow \Diamond\neg P$. In the converse direction, we know from part (b) that there is a D'' such that

$$\emptyset \vdash D'' \rightsquigarrow \Box P \Leftrightarrow \neg\Diamond\neg P.$$

Now let D_2 be the following deduction:

D'' ;
left-iff $\Box P \Leftrightarrow \neg\Diamond\neg P$;
assume $\Diamond\neg P$ **in**
 suppose-absurd $\Box P$ **in**
 begin

modus-ponens $\Box P \Rightarrow \neg \Diamond \neg P, \Box P;$
absurd $\Diamond \neg P, \neg \Diamond \neg P$
end

It follows that

$$\emptyset \vdash D_1; D_2; \text{equivalence } \neg \Box P \Rightarrow \Diamond \neg P, \Diamond \neg P \Rightarrow \neg \Box P \rightsquigarrow \neg \Box P \Leftrightarrow \Diamond \neg P$$

and this establishes part (c). Finally, the following deduction derives $\neg \Diamond P \Leftrightarrow \Box \neg P$ from the empty assumption base:

assume $\neg \Diamond P$ **in**
 begin
 suppose-absurd $\neg \Box \neg P$ **in**
 begin
 \Diamond -intro $\neg \Box \neg P;$
 absurd $\Diamond P, \neg \Diamond P$
 end;
 double-negation $\neg \neg \Box \neg P$
 end;
 assume $\Box \neg P$ **in**
 suppose-absurd $\Diamond P$ **in**
 begin
 \Diamond -elim $\Diamond P;$
 absurd $\Box \neg P, \neg \Box \neg P$
 end;
 equivalence $\neg \Diamond P \Rightarrow \Box \neg P, \Box \neg P \Rightarrow \neg \Diamond P$

This establishes (d) and completes the proof. ■

Next we define a binary relation \equiv on propositions via the rules shown in Figure 1.1. An inspection of the rules will show that this relation obtains between two propositions whenever one can be obtained from the other by zero or more applications of the so-called “modality-interchange rule”,⁵ whereby any subproposition of the form $\neg M_1 \cdots M_n P$ can be replaced by $M'_1 \cdots M'_n \neg P$, where every M_i is either \Box or \Diamond , and $\Box' = \Diamond, \Diamond' = \Box$. Thus a negation sign is moved inwards over a sequence of modal

⁵Hughes and Creswell call this “LMI” for “LM-Interchange”, where they use L for \Box and M for \Diamond .

$\frac{}{P \equiv P} \quad [K_1]$	$\frac{P \equiv Q}{Q \equiv P} \quad [K_2]$	$\frac{P \equiv Q}{P \equiv \neg\neg Q} \quad [K_3]$
$\frac{P \equiv Q}{\circ P \equiv \circ Q} \quad [K_4]$ for $\circ \in \{\neg, \Box, \Diamond\}$	$\frac{P_1 \equiv P'_1 \quad P_2 \equiv P'_2}{P_1 \circ P_2 \equiv P'_1 \circ P'_2} \quad [K_5]$ for $\circ \in \{\wedge, \vee, \Rightarrow, \Leftrightarrow\}$	
$\frac{\neg P \equiv Q}{\neg \Box P \equiv \Diamond Q} \quad [K_6]$	$\frac{\neg P \equiv Q}{\neg \Diamond P \equiv \Box Q} \quad [K_7]$	

Figure 1.1: The definition of the “modality interchange” relation \equiv .

operators, with each such operator being swapped along the way (from \Box to \Diamond and conversely).⁶

For instance, consider the equivalence

$$\neg \Box \Diamond A \wedge \Box \Box B \equiv \Diamond \Box \neg A \wedge \Box \neg \Diamond \neg B \quad (1.9)$$

which, intuitively, holds because it is of the form $P_1 \wedge P_2 \equiv P'_1 \wedge P'_2$, where

- P'_1 (i.e., $\Diamond \Box \neg A$) can be obtained from P_1 (i.e., $\neg \Box \Diamond A$) by moving the negation sign inwards while swapping modal operators along the way; and
- P_2 ($\Box \Box B$) can be obtained from P'_2 ($\Box \neg \Diamond \neg B$) by moving the outer negation sign inwards and switching \Diamond with \Box , resulting in $\Box \Box \neg \neg B$, and then applying double negation within the context of the two necessity operators to get $\Box \Box B$.

Here is a derivation establishing that 1.9 indeed follows from our rules:

- | | |
|---|----------------------|
| 1. $\neg A \equiv \neg A$ | [K ₁] |
| 2. $\neg \Diamond A \equiv \Box \neg A$ | 1, [K ₇] |
| 3. $\neg \Box \Diamond A \equiv \Diamond \Box \neg A$ | 2, [K ₆] |
| 4. $B \equiv B$ | [K ₁] |
| 5. $B \equiv \neg \neg B$ | 4, [K ₃] |

⁶In fact the rules for \equiv allow us to do more than that: we can also eliminate—or introduce—any number of pairs of adjacent negation signs, at any given point. For example, we have $\neg \Diamond \neg \neg \Box P \equiv \Box \Diamond \neg P$, even though these propositions are not interconvertible using modality interchange only.

- | | |
|--|-------------------------|
| 6. $\neg\neg B \equiv B$ | 5, [K ₂] |
| 7. $\neg\diamond\neg B \equiv \Box B$ | 6, [K ₇] |
| 8. $\Box B \equiv \neg\diamond\neg B$ | 7, [K ₂] |
| 9. $\Box\Box B \equiv \Box\neg\diamond\neg B$ | 8, [K ₄] |
| 10. $\neg\Box\diamond A \wedge \Box\Box B \equiv \diamond\Box\neg A \wedge \Box\neg\diamond\neg B$ | 3, 9, [K ₅] |

The next result is very useful: it shows that we can prove $P \Leftrightarrow Q$ in \mathcal{K} whenever $P \equiv Q$.

Lemma 1.6 *If $P \equiv Q$ then $\vdash_{\mathcal{K}} P \Leftrightarrow Q$.*

Proof: By strong induction on the length n of the derivation of the judgment $P \equiv Q$. When $n = 0$ the said judgment must be an instance of axiom [K₁], and the result here is immediate. For $n > 0$ we perform a case analysis on the rule by which the last judgment of the derivation (i.e., $P \Leftrightarrow Q$) was obtained. Rules [K₂ and [K₃] are handled by simple propositional reasoning. For [K₄], the judgment in question must be of the form $\circ P_1 \Leftrightarrow \circ P_2$, for $\circ \in \{\neg, \Box, \diamond\}$, with $P_1 \equiv P_2$ derivable in fewer than n steps. Hence, by the inductive hypothesis, $\vdash_{\mathcal{K}} P_1 \Leftrightarrow P_2$, and the result now follows from Lemma 1.1, or Lemma 1.2, or Lemma 1.3, depending on the identity of \circ . The case of [K₅] is similar, using Lemma 1.1.

In the case of [K₆], the judgment must be of the form $\neg\Box P \equiv \diamond Q$, where $\neg P \equiv Q$ is derivable in fewer than n steps. Hence, inductively, we have $\vdash_{\mathcal{K}} \neg P \Leftrightarrow Q$. By Lemma 1.3, this yields

$$\vdash_{\mathcal{K}} \diamond\neg P \Leftrightarrow \diamond Q. \quad (1.10)$$

But part (c) of Lemma 1.5 gives

$$\vdash_{\mathcal{K}} \neg\Box P \Leftrightarrow \diamond\neg P \quad (1.11)$$

and now

$$\vdash_{\mathcal{K}} \neg\Box P \Leftrightarrow \diamond Q$$

follows from 1.11 and 1.10 by virtue of the fact that $\vdash_{\mathcal{K}} P_1 \Leftrightarrow P_3$ whenever $\vdash_{\mathcal{K}} P_1 \Leftrightarrow P_2$ and $\vdash_{\mathcal{K}} P_2 \Leftrightarrow P_3$ (this is easily proved by propositional reasoning).

The last possibility, [K₇], is treated similarly: the inductive hypothesis yields $\vdash_{\mathcal{K}} \neg P \Leftrightarrow Q$. Applying Lemma 1.2 to that results in

$$\vdash_{\mathcal{K}} \Box\neg P \Leftrightarrow \Box Q. \quad (1.12)$$

But Lemma 1.5 gives

$$\vdash_{\mathcal{K}} \neg\diamond P \Leftrightarrow \Box\neg P \quad (1.13)$$

and now the desired $\vdash_{\mathcal{K}} \neg\diamond P \Leftrightarrow \Box Q$ follows from 1.13 and 1.12. ■

Later on we will show an interesting recursive *method* that will take any given P , Q , and if in fact $P \equiv Q$ holds, it will *prove* the equivalence $P \Leftrightarrow Q$ automatically.

Next we define a mapping γ from propositions to propositions as follows:

$$\begin{aligned}\gamma(\Box P) &= \neg\Diamond\neg\gamma(P) \\ \gamma(P) &= \text{atom}?(P) \rightarrow P, \mathcal{F}(P, \gamma)\end{aligned}$$

Thus $\gamma(P)$ is obtained from P by removing every occurrence of the necessity operator \Box , substituting $\neg\Diamond\neg$ in its place. For instance,

$$\gamma(\Diamond\Box\neg P \Rightarrow \neg\Box Q) = \Diamond\neg\Diamond\neg\neg P \Rightarrow \neg\neg\Diamond\neg Q.$$

We also define a dual mapping δ that removes possibility operators:

$$\begin{aligned}\delta(\Diamond P) &= \neg\Box\neg\delta(P) \\ \delta(P) &= \text{atom}?(P) \rightarrow P, \mathcal{F}(P, \delta).\end{aligned}$$

The following lemma shows that in \mathcal{K} we can prove any proposition P to be equivalent to $\gamma(P)$ or $\delta(P)$:

Lemma 1.7 $\vdash_{\mathcal{K}} P \Leftrightarrow \gamma(P)$, $\vdash_{\mathcal{K}} P \Leftrightarrow \delta(P)$.

Proof: We need only show that $P \equiv \gamma(P)$; the result will then follow from Lemma 1.6. A straightforward induction on the structure of P will achieve this. When P is an atom the result is immediate by $[K_1]$. When P is of the form $\neg Q$, the inductive hypothesis gives $Q \equiv \gamma(Q)$, so $[K_4]$ yields $\neg Q \equiv \neg\gamma(Q)$, which is to say $P \equiv \gamma(P)$. When P is of the form $\Diamond Q$ or $P_1 \circ P_2$ for some binary propositional constructor \circ , the result again follows from the inductive hypothesis and rules $[K_4]$ and $[K_5]$. Finally, when P is of the form $\Box Q$ we have:

1. $Q \equiv \gamma(Q)$ From the inductive hypothesis
2. $Q \equiv \neg\neg\gamma(Q)$ 1, $[K_3]$
3. $\neg\neg\gamma(Q) \equiv Q$ 2, $[K_2]$
4. $\neg\Diamond\neg\gamma(Q) \equiv \Box Q$ 3, $[K_7]$
5. $\Box Q \equiv \neg\Diamond\neg\gamma(Q)$ 4, $[K_2]$

which is to say $P \equiv \gamma(P)$. The proof for δ is analogous. ■

1.2 Kripke semantics

By a *frame* we will mean an ordered pair $\mathcal{F} = \langle W, R \rangle$ consisting of an arbitrary set W and a binary relation $R \subseteq W^2$. We will refer to the elements of W as the *worlds* of \mathcal{F} (or as “ \mathcal{F} -worlds”), and to R as the *accessibility relation* on these worlds. Accordingly, when $w_1 R w_2$ we will say that w_2 is “accessible” from w_1 , or that w_1 can “see” w_2 . When we wish to make the relationship between W and \mathcal{F} explicit, we will write $W_{\mathcal{F}}$ instead of just W ; and likewise with $R_{\mathcal{F}}$.

By an *atomic assignment* (or *interpretation*) on \mathcal{F} we will mean a function ρ from the set of all atomic propositions to the power-set of $W_{\mathcal{F}}$. Thus an assignment on \mathcal{F} is a function that assigns a unique set of \mathcal{F} -worlds to each atomic proposition. Intuitively, these will be the worlds in which that atomic proposition is to count as true. We will write $\mathcal{A}_{\mathcal{F}}$ for the set of all assignments on \mathcal{F} .

Now let a frame $\mathcal{F} = (W, R)$ be given, let ρ be an assignment on \mathcal{F} , and consider an arbitrary proposition P . We write $w \models_{\rho} P$ to signify that w *satisfies* P with respect to ρ , or that P *holds* in w with respect to ρ . This satisfaction relation is defined by structural induction on P :

- $w \models_{\rho}$ **true**, $w \not\models_{\rho}$ **false**.
- If P is an atom A , then $w \models_{\rho} P$ iff $w \in \rho(A)$.
- If P is a negation $\neg Q$, then $w \models_{\rho} P$ iff $w \not\models_{\rho} Q$.
- If P is a conjunction $P_1 \wedge P_2$, then $w \models_{\rho} P$ iff both $w \models_{\rho} P_1$ and $w \models_{\rho} P_2$.
- If P is a disjunction $P_1 \vee P_2$, then $w \models_{\rho} P$ iff $w \models_{\rho} P_1$ or $w \models_{\rho} P_2$.
- If P is a conditional $P_1 \Rightarrow P_2$, then $w \models_{\rho} P$ iff $w \models_{\rho} P_2$ whenever $w \models_{\rho} P_1$.
- If P is a biconditional $P_1 \Leftrightarrow P_2$, then $w \models_{\rho} P$ iff $w \models_{\rho} P_2$ whenever $w \models_{\rho} P_1$, and $w \models_{\rho} P_1$ whenever $w \models_{\rho} P_2$.
- If P is of the form $\Box Q$, then $w \models_{\rho} P$ iff $w' \models P$ for every w' such that $w R w'$.
- If P is of the form $\Diamond Q$, then $w \models_{\rho} P$ iff $w' \models P$ for some w' such that $w R w'$.

For a set of propositions Φ , we write $w \models_{\rho} \Phi$ to mean that $w \models_{\rho} P$ for every $P \in \Phi$. Thus, in particular, we always have $w \models_{\rho} \emptyset$.

We say that a frame \mathcal{F} *satisfies* P iff $w \models_{\rho} P$ for some $w \in W_{\mathcal{F}}$ and $\rho \in \mathcal{A}_{\mathcal{F}}$. We indicate this by writing $\mathcal{F} \models_{w, \rho} P$. We call P *satisfiable* or *unsatisfiable* in a frame \mathcal{F} according to whether or not \mathcal{F} satisfies P , respectively. We say that P is *valid* in \mathcal{F} , or that \mathcal{F} *validates* (or is a *model* of) P , iff $\mathcal{F} \models_{w, \rho} P$ for every $w \in W_{\mathcal{F}}$ and $\rho \in \mathcal{A}_{\mathcal{F}}$.

We denote this by writing $\mathcal{F} \models P$. For a set of propositions Φ , we write $\mathcal{F} \models \Phi$ to mean that $\mathcal{F} \models P$ for every $P \in \Phi$. Thus, vacuously, $\mathcal{F} \models \emptyset$ for all \mathcal{F} . We introduce a certain kind of logical implication between sets of propositions and single propositions as follows: we define $\Phi \models^{\mathcal{F}} P$ to hold iff for every $w \in W_{\mathcal{F}}$ and $\rho \in \mathcal{A}_{\mathcal{F}}$, we have $w \models_{\rho} P$ whenever $w \models_{\rho} \Phi$. When Φ is a singleton $\{Q\}$, we might write $Q \models^{\mathcal{F}} P$ instead of $\{Q\} \models^{\mathcal{F}} P$.

Finally, let \mathbf{C} be a set of frames. We say that a proposition P is valid in \mathbf{C} , written $\mathbf{C} \models P$, iff $\mathcal{F} \models P$ for every $\mathcal{F} \in \mathbf{C}$. That is, P is valid in \mathbf{C} iff P is valid in every frame of \mathbf{C} . We say that \mathcal{K} is *sound* in a class of frames \mathbf{C} iff $\mathbf{C} \models P$ whenever $\vdash_{\mathcal{K}} P$. In other words, \mathcal{K} is sound in a given set of frames \mathbf{C} iff every proposition that can be derived from the empty assumption base is valid in every frame of \mathbf{C} . And we say that \mathcal{K} is *complete* in \mathbf{C} iff $\vdash_{\mathcal{K}} P$ whenever $\mathbf{C} \models P$; i.e., if P is valid in \mathbf{C} , then it can be derived from the empty assumption base.

Lemma 1.8 *For all frames \mathcal{F} , $\Box(P \Rightarrow Q) \models^{\mathcal{F}} \Box P \Rightarrow \Box Q$.*

Proof: Pick any $w \in W_{\mathcal{F}}$ and $\rho \in \mathcal{A}_{\mathcal{F}}$, and suppose that

$$w \models_{\rho} \Box(P \Rightarrow Q). \quad (1.14)$$

We need to show that $w \models_{\rho} \Box P \Rightarrow \Box Q$, i.e., that $w \models_{\rho} \Box Q$ whenever $w \models_{\rho} \Box P$. Accordingly, suppose that

$$w \models_{\rho} \Box P. \quad (1.15)$$

Now to show the desired $w \models_{\rho} \Box Q$, we need to prove that $w' \models_{\rho} Q$ for every world w' that is accessible from w (i.e., such that $w R_{\mathcal{F}} w'$). To that end, consider any such w' . From 1.14, we infer that $w' \models_{\rho} P \Rightarrow Q$, while from 1.15 we conclude $w' \models_{\rho} P$. But then by the definition of the satisfaction relation we infer $w' \models_{\rho} Q$. Since this was proven for *any* w' that is accessible from w , we may conclude that $w \models_{\rho} \Box Q$. ■

Lemma 1.9 *For all frames \mathcal{F} :*

- (a) *If $\Phi \cup \{P\} \models^{\mathcal{F}} Q$ then $\Phi \models^{\mathcal{F}} P \Rightarrow Q$.*
- (b) *If $\Phi \models^{\mathcal{F}} P$ and $\Phi \cup \{P\} \models^{\mathcal{F}} Q$ then $\Phi \models^{\mathcal{F}} Q$.*

Proof: For (a), consider any \mathcal{F} , $w \in W_{\mathcal{F}}$, and $\rho \in \mathcal{A}_{\mathcal{F}}$, and suppose that

$$w \models_{\rho} \Phi. \quad (1.16)$$

We must show that $w \models_\rho P \Rightarrow Q$, i.e., that $w \models_\rho Q$ on the assumption that $w \models_\rho P$. Accordingly, suppose that

$$w \models_\rho P. \quad (1.17)$$

Then 1.16 and 1.17 yield $w \models_\rho \Phi \cup \{P\}$, and now the supposition $\Phi \cup \{P\} \models^\mathcal{F} Q$ yields the desired $w \models_\rho Q$. Part (b) is immediate. ■

Corollary 1.10 *The proposition $\Box(P \Rightarrow Q) \Rightarrow (\Box P \Rightarrow \Box Q)$ is valid in every frame.*

Proof: Consider any frame \mathcal{F} . From Lemma 1.8 and part (a) of Lemma 1.9 we deduce $\emptyset \models^\mathcal{F} \Box(P \Rightarrow Q) \Rightarrow (\Box P \Rightarrow \Box Q)$, which is to say $w \models_\rho \Box(P \Rightarrow Q) \Rightarrow (\Box P \Rightarrow \Box Q)$ for every $w \in W_\mathcal{F}$ and $\rho \in \mathcal{A}_\mathcal{F}$. ■

Lemma 1.11 *If $\beta \vdash D \rightsquigarrow P$ then $\beta \models^\mathcal{F} P$ for all frames \mathcal{F} .*

Proof: By induction on the structure of D . When D is either the claim **true** or the claim **¬false** the result is immediate. When D is some other claim P then the assumption $\beta \vdash D \rightsquigarrow P$ means that β contains P , and hence $\beta \models^\mathcal{F} P$ is trivial. When D is a primitive deduction then the assumption $\beta \vdash D \rightsquigarrow P$ means that the judgment $\beta \vdash D \rightsquigarrow P$ is either an instance of one of the axioms shown in Figure ?? or an instance of 1.4. In the latter case the result follows from Lemma 1.8.

In the former case the result is readily verified by a case analysis of the propositional primitive rules. We illustrate with **modus-tollens**, in which case the judgment $\beta \vdash D \rightsquigarrow P$ must be of the form

$$\beta' \cup \{P_1 \Rightarrow P_2, \neg P_2\} \vdash \text{modus-tollens } P_1 \Rightarrow P_2, \neg P_2 \rightsquigarrow \neg P_1.$$

Pick any frame \mathcal{F} , $w \in W_\mathcal{F}$, and $\rho \in \mathcal{A}_\mathcal{F}$, and suppose that $w \models_\rho \beta' \cup \{P_1 \Rightarrow P_2, \neg P_2\}$, so that $w \models_\rho P_1 \Rightarrow P_2$ and $w \models_\rho \neg P_2$. Suppose, by way of contradiction, that $w \models_\rho P_1$. Then, from the assumption $w \models_\rho P_1 \Rightarrow P_2$ and the definition of the satisfaction relation we get $w \models_\rho P_2$, which contradicts the assumption $w \models_\rho \neg P_2$. Accordingly, we must have $w \not\models_\rho P_1$, which is to say $w \models_\rho \neg P_1$. Since this argument was given for arbitrary w and ρ , we conclude that $\beta' \cup \{P_1 \Rightarrow P_2, \neg P_2\} \models^\mathcal{F} \neg P_1$.

When D is a hypothetical deduction **assume P_1 in D_b** , the conclusion P must be of the form $P_1 \Rightarrow P_2$, where

$$\beta \cup \{P_1\} \vdash D_b \rightsquigarrow P_2 \quad (1.18)$$

and we need to show that

$$\beta \models^\mathcal{F} P_1 \Rightarrow P_2 \quad (1.19)$$

for an arbitrary frame \mathcal{F} . Fixing such a frame, 1.18 and the inductive hypothesis yield $\beta \cup \{P_1\} \models^\mathcal{F} P_2$, and now 1.19 follows from part (a) of Lemma 1.9.

Next, suppose that D is of the form $D_1; D_2$. Then we must have

$$\beta \vdash D_1 \rightsquigarrow P_1 \tag{1.20}$$

and

$$\beta \cup \{P_1\} \vdash D_2 \rightsquigarrow P_2 \tag{1.21}$$

and we need to show that $\beta \models^{\mathcal{F}} P_2$ for any \mathcal{F} . This follows from part (b) of Lemma 1.9 by applying the inductive hypothesis to 1.20 and 1.21.

Finally, suppose that D is a necessitation **nec** : D_b . Then the conclusion P must be of the form $\Box Q$, where

$$\emptyset \vdash D_b \rightsquigarrow Q \tag{1.22}$$

and thus we need to show that $\beta \models^{\mathcal{F}} \Box Q$. Pick any frame \mathcal{F} and $w \in W_{\mathcal{F}}$, $\rho \in \mathcal{A}_{\mathcal{F}}$. We will prove that $w \models_{\rho} \Box Q$ regardless of whether or not $w \models_{\rho} \beta$. The proof is simple: 1.22 and the inductive hypothesis imply that $w' \models_{\rho} Q$ for all w' , and thus for all w' such that $w R_{\mathcal{F}} w'$ —hence $w \models_{\rho} \Box Q$. ■

Corollary 1.12 (Soundness of \mathcal{K}) *If $\vdash_{\mathcal{K}} P$ then P is valid in every frame.*

Proof: $\vdash_{\mathcal{K}} P$ means that there is a D such that $\emptyset \vdash D \rightsquigarrow P$, and thus the preceding result entails that for any frame \mathcal{F} , we have $w \models_{\rho} P$ for all $w \in W_{\mathcal{F}}$ and $\rho \in \mathcal{A}_{\mathcal{F}}$ —which is to say that P is valid in \mathcal{F} . ■

Corollary 1.13 *For all frames \mathcal{F} , $\gamma(P) \models^{\mathcal{F}} P$ and $P \models^{\mathcal{F}} \gamma(P)$. Likewise, $\delta(P) \models^{\mathcal{F}} P$ and $P \models^{\mathcal{F}} \delta(P)$.*

Proof: From Lemma 1.11, since, by Lemma 1.7, we can derive P and $\gamma(P)$ from each other, and likewise for P and $\delta(P)$. ■

1.3 Standard modal DPLs

By a *standard* modal DPL \mathcal{S} we will understand a DPL whose syntax and evaluation semantics are just as in \mathcal{K} , the only possible exception being that \mathcal{S} might have additional primitive rules (which we will call *\mathcal{S} -specific rules*) and evaluation axioms that specify the semantics of those rules. Formally, an evaluation axiom for a rule R of arity n can be viewed as a set E_R of $(n+2)$ -tuples $\langle \beta, P_1, \dots, P_n, P \rangle$, where β is an assumption base and P_1, \dots, P_n, P are propositions, such that:⁷

⁷The first requirement preserves conclusion uniqueness, and is similar to requirement **PM1** in the $\lambda\mu$ calculus ([1]). The second requirement preserves the dilution property of $\mathcal{CN}\mathcal{D}$. In the $\lambda\mu$ calculus there is no need to impose an analogous stipulation on primitive methods because dilution is ensured by fiat via [R10] (Figure ??).

1. $P = P'$ whenever $\langle \beta, P_1, \dots, P_n, P \rangle \in E_R$ and $\langle \beta', P_1, \dots, P_n, P' \rangle \in E_R$; and
2. $\langle \beta \cup \beta', P_1, \dots, P_n, P \rangle \in E_R$ whenever $\langle \beta, P_1, \dots, P_n, P \rangle \in E_R$.

For each such tuple in E_R , we stipulate that

$$\beta \vdash R P_1, \dots, P_n \rightsquigarrow P.$$

Then $\vdash_{\mathcal{S}} \beta \vdash D \rightsquigarrow P$ is defined as usual: iff there exists a sequence

$$\beta_1 \vdash D_1 \rightsquigarrow P_1, \dots, \beta_n \vdash D_n \rightsquigarrow P_n$$

such that $\beta_n = \beta, D_n = D, P_n = P$, and for each $i = 1, \dots, n$, the judgment $\beta_i \vdash D_i \rightsquigarrow P_i$ is either an instance of one of the axioms $[R_1]$, $[R_2]$, or $[R_3]$; or an instance of one of the the axioms in Figure ?? or of 1.4, or an instance of an axiom for a \mathcal{S} -specific rule;⁸ or else follows from previous judgments through $[R_4]$, or $[R_5]$, or $[R_6]$. When \mathcal{S} is understood or immaterial, we may simply write $\beta \vdash D \rightsquigarrow P$ as an abbreviation for $\vdash_{\mathcal{S}} \beta \vdash D \rightsquigarrow P$. Finally, for any set of propositions Φ , we write $\Phi \vdash_{\mathcal{S}} P$ iff there is a finite subset $\beta \subseteq \Phi$ and a D such that $\vdash_{\mathcal{S}} \beta \vdash D \rightsquigarrow P$.

Clearly, all of the modal DPLs we have considered so far (\mathcal{K} , \mathcal{D} , \mathcal{T} , \mathcal{S}_4 , and \mathcal{S}_5) are standard. For the remainder of this section, \mathcal{S} should be understood to stand for an arbitrary standard modal DPL; and all judgements of the form $\beta \vdash D \rightsquigarrow P$ should be understood as stating $\vdash_{\mathcal{S}} \beta \vdash D \rightsquigarrow P$ for such \mathcal{S} . The following results carry over from $\mathcal{CN}\mathcal{D}$:

Theorem 1.14 (Uniqueness) *If $\beta_1 \vdash D \rightsquigarrow P_1$ and $\beta_2 \vdash D \rightsquigarrow P_2$ then $P_1 = P_2$.*

Lemma 1.15 (Reflexivity) $\Phi \vdash_{\mathcal{S}} P$ for all $P \in \Phi$.

Lemma 1.16 (Monotonicity) *If $\Phi \vdash_{\mathcal{S}} P$ then $\Phi \cup \Psi \vdash_{\mathcal{S}} P$.*

Lemma 1.17 (Dilution) *If $\beta \vdash D \rightsquigarrow P$ then $\beta \cup \beta' \vdash D \rightsquigarrow P$.*

Lemma 1.18 (a) *If $\beta \cup \{P\} \vdash D \rightsquigarrow Q$ then $\beta \vdash \mathbf{assume} P \text{ in } D \rightsquigarrow P \Rightarrow Q$; (b) if $\beta \cup \{P\} \vdash D \rightsquigarrow \mathbf{false}$ then $\beta \vdash \mathbf{suppose-absurd} P \text{ in } D \rightsquigarrow \neg P$; and (c) $\Phi \vdash_{\mathcal{S}} P \Rightarrow Q$ iff $\Phi \cup \{P\} \vdash_{\mathcal{S}} Q$.*

Lemma 1.19 *For all Φ and P :*

⁸More formally, using the notation introduced in footnote ??: if D is a primitive rule application of the form $R P_1, \dots, P_k$, where $\langle \beta_i, P_1, \dots, P_k, P \rangle \in E_R$.

- (a) $\Phi \vdash_S P$ iff $\Phi \vdash_S \neg\neg P$
- (b) $\Phi \vdash_S P \wedge Q$ iff $\Phi \vdash_S P$ and $\Phi \vdash_S Q$
- (c) if $\Phi \vdash_S P$ or $\Phi \vdash_S Q$ then $\Phi \vdash_S P \vee Q$
- (d) if $\Phi \vdash_S P \Rightarrow Q$ and $\Phi \vdash_S P$ then $\Phi \vdash_S Q$
- (e) $\Phi \vdash_S P \Leftrightarrow Q$ iff $\Phi \vdash_S P \Rightarrow Q$ and $\Phi \vdash_S Q \Rightarrow P$
- (f) if $\Phi \vdash_S \neg P$ and $\Phi \vdash_S \neg Q$ then $\Phi \vdash_S \neg(P \vee Q)$.

The following lemmas will come handy later:

Lemma 1.20 *If $\Phi \vdash_S \Box(P_1 \Rightarrow \dots \Rightarrow P_n \Rightarrow P)$ then*

$$\Phi \vdash_S \Box P_1 \Rightarrow \dots \Rightarrow \Box P_n \Rightarrow \Box P$$

for every $n > 0$.

Proof: By induction on n . When $n = 1$, the assumption

$$\Phi \vdash_S \Box(P_1 \Rightarrow \dots \Rightarrow P_n \Rightarrow P) \tag{1.23}$$

gives $\Phi \vdash_S \Box P_1 \Rightarrow \Box P$ directly via **K**. When $n > 1$, assumption 1.23 gives

$$\Phi \vdash_S \Box P_1 \Rightarrow \Box(P_2 \Rightarrow \dots \Rightarrow P_n \Rightarrow P) \tag{1.24}$$

again via **K**. Now by 1.24 and Lemma 1.18 we get

$$\Phi \cup \{\Box P_1\} \vdash_S \Box(P_2 \Rightarrow \dots \Rightarrow P_n \Rightarrow P). \tag{1.25}$$

Now from 1.25 and the inductive hypothesis we obtain

$$\Phi \cup \{\Box P_1\} \vdash_S \Box P_2 \Rightarrow \dots \Rightarrow \Box P_n \Rightarrow \Box P$$

and thus Lemma 1.18 yields $\Phi \vdash_S \Box P_1 \Rightarrow \Box P_2 \Rightarrow \dots \Rightarrow \Box P_n \Rightarrow \Box P$, completing the induction. ■

Lemma 1.21 *If $\{P_1, \dots, P_n\} \vdash_S P$ then $\{\Box P_1, \dots, \Box P_n\} \vdash_S \Box P$.*

Proof: By repeated applications of Lemma 1.18, the assumption $\{P_1, \dots, P_n\} \vdash_S P$ yields

$$\emptyset \vdash_S P_1 \Rightarrow \dots \Rightarrow P_n \Rightarrow P$$

which means that there is a D such that $\emptyset \vdash D \rightsquigarrow P_1 \Rightarrow \dots \Rightarrow P_n \Rightarrow P$. Therefore,

$$\emptyset \vdash \mathbf{nec}: D \rightsquigarrow \Box(P_1 \Rightarrow \dots \Rightarrow P_n \Rightarrow P)$$

$$\begin{array}{l}
Eval(D, \beta) = ev(D) \\
\text{where} \\
ev(P) = P \in \beta \cup \{\mathbf{true}, \neg\mathbf{false}\} \rightarrow P, \text{ error} \\
ev(\text{Prim-Rule } P_1, \dots, P_n) = do\text{-prim-rule}(\text{Prim-Rule}, [P_1, \dots, P_n], \beta) \\
ev(\mathbf{assume } P \text{ in } D) = \text{let } Q = Eval(D, \beta \cup \{P\}) \\
\quad \text{in} \\
\quad \quad Q = \text{error} \rightarrow \text{error}, P \Rightarrow Q \\
ev(D_1; D_2) = \text{let } P_1 = Eval(D_1, \beta) \\
\quad \text{in} \\
\quad \quad P_1 = \text{error} \rightarrow \text{error}, Eval(D_2, \beta \cup \{P_1\}) \\
ev(\mathbf{nec}: D) = \text{let } P = Eval(D, \emptyset) \\
\quad \text{in} \\
\quad \quad P = \text{error} \rightarrow \text{error}, \Box P
\end{array}$$

Figure 1.2: An interpreter for any normal modal DPL.

and, by Lemma 1.20, $\emptyset \vdash_{\mathcal{S}} \Box P_1 \Rightarrow \dots \Rightarrow \Box P_n \Rightarrow \Box P$. Finally, again by repeated applications of Lemma 1.18, we obtain $\{\Box P_1, \dots, \Box P_n\} \vdash_{\mathcal{S}} \Box P$. ■

An interpreter for standard modal DPLs is easily obtained by extending the $\mathcal{CN}\mathcal{D}$ interpreter $Eval$ to account for necessitation (of course we also need to adjust the auxiliary function $do\text{-prim-rule}$ ⁹); such an extension is shown in Figure 1.2. Its correctness and termination are likewise established by a straightforward induction on a given D :

Theorem 1.22 (Termination and correctness of the modal interpreter) *The algorithm $Eval$ always terminates. In particular, the recursion tree spawned by a call $Eval(D, \beta)$ has size $\Theta(n)$, where n is the size of D . Accordingly, a deduction is evaluated in time linear in its size. Moreover, for all P and D ,*

$$\beta \vdash D \rightsquigarrow P \text{ iff } Eval(D, \beta) = P.$$

Hence, by termination, $Eval(D, \beta) = \text{error}$ iff there is no P such that $\beta \vdash D \rightsquigarrow P$.

Next, we say that a set of propositions Φ is \mathcal{S} -inconsistent iff $\Phi \vdash_{\mathcal{S}} \mathbf{false}$, and \mathcal{S} -consistent otherwise. We call Φ maximally \mathcal{S} -consistent iff it is \mathcal{S} -consistent and not

⁹Strictly speaking, to ensure this we also need to require that every \mathcal{S} -specific rule R is computable, meaning that there is an algorithm which, given a β and arguments P_1, \dots, P_n , will determine if there is a P such that $\langle \beta, P_1, \dots, P_n, P \rangle \in E_R$, and if so, what that P is (a proviso akin to **PM2** in the $\lambda\mu$ calculus); we will tacitly understand this to be the case in all standard modal DPLs.

properly contained in any \mathcal{S} -consistent set. This means that for every P that is not a member of Φ , the set $\Phi \cup \{P\}$ is \mathcal{S} -inconsistent. The proofs of the following results carry over directly from the classical propositional case, save the obvious differences that propositions are now generated by 1.1 and that consistency is understood as \mathcal{S} -consistency:

Lemma 1.23 Φ is \mathcal{S} -inconsistent iff $\Phi \vdash_{\mathcal{S}} P$ for every proposition P , i.e., iff “everything is provable from Φ ”. Alternatively, Φ is \mathcal{S} -inconsistent iff there is a P such that $\Phi \vdash_{\mathcal{S}} P$ and $\Phi \vdash_{\mathcal{S}} \neg P$.

Lemma 1.24 If $\Phi \cup \{P\} \vdash_{\mathcal{S}} Q$ and $\Phi \cup \{P\} \vdash_{\mathcal{S}} \neg Q$ then $\Phi \vdash_{\mathcal{S}} \neg P$.

Lemma 1.25 If $\Phi \cup \{P\} \vdash_{\mathcal{S}} Q$ and $\Phi \cup \{P\} \vdash_{\mathcal{S}} \neg Q$ then $\Phi \vdash_{\mathcal{S}} \neg P$.

Lemma 1.26 $\Phi \vdash_{\mathcal{S}} P$ iff $\Phi \cup \{\neg P\}$ is \mathcal{S} -inconsistent. Equivalently, $\Phi \vdash_{\mathcal{S}} \neg P$ iff $\Phi \cup \{P\}$ is \mathcal{S} -inconsistent.

Lemma 1.27 Maximally \mathcal{S} -consistent sets are deductively closed, i.e., if Φ is maximally \mathcal{S} -consistent and $\Phi \vdash_{\mathcal{S}} P$ then $P \in \Phi$.

Lemma 1.28 Maximally \mathcal{S} -consistent sets are saturated. That is, if Φ is maximally \mathcal{S} -consistent then for all P and Q we have: (a) $P \in \Phi$ iff $\neg P \notin \Phi$; (b) $P \wedge Q \in \Phi$ iff both $P \in \Phi$ and $Q \in \Phi$; (c) $P \vee Q \in \Phi$ iff $P \in \Phi$ or $Q \in \Phi$; (d) $P \Rightarrow Q \in \Phi$ iff $Q \in \Phi$ whenever $P \in \Phi$; and (e) $P \Leftrightarrow Q \in \Phi$ iff $P \in \Phi$ iff $Q \in \Phi$.

Lemma 1.29 Every \mathcal{S} -consistent set of propositions is contained in some maximally \mathcal{S} -consistent set of propositions.

The next lemma is the only result we will need about \mathcal{S} -consistency that is not inherited directly from the propositional case:

Lemma 1.30 If Φ is \mathcal{S} -consistent and $\diamond Q \in \Phi$ then $\{P \mid \Box P \in \Phi\} \cup \{Q\}$ is \mathcal{S} -consistent.

Proof: By contradiction: supposing that $\{P \mid \Box P \in \Phi\} \cup \{Q\}$ is \mathcal{S} -inconsistent, Lemma 1.26 gives $\{P \mid \Box P \in \Phi\} \vdash_{\mathcal{S}} \neg Q$. Hence, by Lemma 1.21,

$$\{\Box P \mid \Box P \in \Phi\} \vdash_{\mathcal{S}} \Box \neg Q. \quad (1.26)$$

But $\Box \neg Q \equiv \neg \diamond Q$, hence, by Lemma 1.6, part (e) of 1.19, and monotonicity, we get

$$\{\Box P \mid \Box P \in \Phi\} \vdash_{\mathcal{S}} \Box \neg Q \Rightarrow \neg \diamond Q \quad (1.27)$$

and now from 1.26, 1.27, and part (d) of 1.19, we infer $\{\Box P \mid \Box P \in \Phi\} \vdash_{\mathcal{S}} \neg\Diamond Q$. But then Lemma 1.26 would imply that

$$\{\Box P \mid \Box P \in \Phi\} \cup \{\Diamond Q\}$$

is \mathcal{S} -inconsistent, and since Φ is a superset of this set, it would have to be \mathcal{S} -inconsistent as well—contradicting our hypothesis. ■

1.3.1 Canonical Kripke frames for standard modal DPLs

Let \mathcal{S} be a standard modal DPL. We define the *canonical \mathcal{S} -frame*

$$\mathcal{F}_{\mathcal{S}} = \langle W_{\mathcal{S}}, R_{\mathcal{S}} \rangle$$

as follows:

- $W_{\mathcal{S}}$ is the set of all maximally \mathcal{S} -consistent sets of propositions; and
- for any $\Phi_1, \Phi_2 \in W_{\mathcal{S}}$, we have $\Phi_1 R_{\mathcal{S}} \Phi_2$ iff $P \in \Phi_2$ whenever $\Box P \in \Phi_1$.

In addition, we define the *canonical \mathcal{S} -interpretation* $\rho_{\mathcal{S}} \in \mathcal{A}_{\mathcal{F}_{\mathcal{S}}}$ as follows: for any atom A ,

$$\Phi \in \rho_{\mathcal{S}}(A) \text{ iff } A \in \Phi.$$

In words, $\rho_{\mathcal{S}}$ dictates that an atom A is true in a world Φ iff $A \in \Phi$. The main result of this section is Theorem 1.32 below. We first state and prove a little fact that will prove handy later on:

Lemma 1.31 *Every world in a canonical \mathcal{S} -frame contains infinitely many propositions of the form $\Box P$.*

Proof: Pick any $\Phi \in W_{\mathcal{S}}$ and let P be any proposition whatsoever. We have

$$\emptyset \vdash \text{nec: assume } P \text{ in } P \rightsquigarrow P \Rightarrow P$$

thus $\emptyset \vdash_{\mathcal{S}} \Box(P \Rightarrow P)$, and by monotonicity, $\Phi \vdash_{\mathcal{S}} \Box(P \Rightarrow P)$. But Φ is deductively closed (being maximally \mathcal{S} -consistent, Lemma 1.27), hence $\Box(P \Rightarrow P) \in \Phi$. (And clearly there is nothing special about propositions of the form $P \Rightarrow P$; by the same argument, we have $\Box Q \in \Phi$ for every propositional tautology Q .) ■

Theorem 1.32 *Let \mathcal{S} be any standard modal DPL and let Φ be any world in the canonical \mathcal{S} -frame. Then, for all P , $\Phi \models_{\rho_{\mathcal{S}}} P$ iff $P \in \Phi$.*

Proof: We will first prove the result for all propositions P that do not contain any occurrences of \diamond . We proceed by induction on the structure of P . When P is **true** or **false** the result is immediate. When P is an atom the result follows from the definition of ρ_S . When P is a negation $\neg Q$ we have

$$\begin{aligned} \Phi &\models_{\rho_S} \neg Q \text{ iff} \\ \Phi &\not\models_{\rho_S} Q \text{ iff} && \text{(by the inductive hypothesis)} \\ Q &\notin \Phi \text{ iff} && \text{(by Lemma 1.28)} \\ \neg Q &\in \Phi. \end{aligned}$$

When P is a conjunction $P_1 \wedge P_2$:

$$\begin{aligned} \Phi &\models_{\rho_S} P_1 \wedge P_2 && \text{iff} \\ \Phi &\models_{\rho_S} P_1 \text{ and } \Phi \models_{\rho_S} P_2 && \text{iff} && \text{(by the inductive hypothesis)} \\ P_1 &\in \Phi \text{ and } P_2 \in \Phi && \text{iff} && \text{(by Lemma 1.28)} \\ P_1 &\wedge P_2 \in \Phi. \end{aligned}$$

The rest of the propositional cases (disjunctions, conditionals, and biconditionals) are handled similarly.

Finally, suppose that P is of the form $\Box Q$. In one direction, suppose that

$$\Box Q \in \Phi. \tag{1.28}$$

Pick any Ψ such that $\Phi R_S \Psi$. From 1.28 and the definition of R_S , $Q \in \Psi$, and therefore, by the inductive hypothesis, $\Psi \models_{\rho_S} Q$. This shows that $\Phi \models_{\rho_S} \Box Q$.

Conversely, suppose that

$$\Phi \models_{\rho_S} \Box Q. \tag{1.29}$$

Set

$$\Omega = \{P \mid \Box P \in \Phi\}.$$

We claim that $\Omega \cup \{\neg Q\}$ is \mathcal{S} -inconsistent. We will prove this by way of contradiction: if $\Omega \cup \{\neg Q\}$ were \mathcal{S} -consistent then, by Lemma 1.29, we would have

$$\Omega \cup \{\neg Q\} \subseteq \Omega^* \tag{1.30}$$

for some maximally \mathcal{S} -consistent Ω^* . By the definition of Ω and 1.30 we get

$$\Phi R_S \Omega^*. \tag{1.31}$$

But 1.31 and 1.29 entail $\Omega^* \models_{\rho_S} Q$, so, by the inductive hypothesis,

$$Q \in \Omega^*. \quad (1.32)$$

But 1.30 and 1.32 are contradictory: they entail $\{Q, \neg Q\} \subseteq \Omega^*$, which implies that Ω^* is \mathcal{S} -inconsistent—a contradiction.

Now since $\Omega \cup \{\neg Q\}$ is \mathcal{S} -inconsistent, Lemma 1.26 implies that there is a finite subset

$$\{P_1, \dots, P_n\} \subseteq \Omega \quad (1.33)$$

and a D such that $\{P_1, \dots, P_n\} \vdash D \rightsquigarrow P$. Therefore, Lemma 1.21 yields

$$\{\Box P_1, \dots, \Box P_n\} \vdash_{\mathcal{S}} \Box Q. \quad (1.34)$$

But by construction, 1.33 entails $\{\Box P_1, \dots, \Box P_n\} \subseteq \Phi$, so 1.34 and monotonicity yield $\Phi \vdash_{\mathcal{S}} \Box Q$, and since Φ is deductively closed (being maximally consistent), $\Box Q \in \Phi$. This completes the induction.¹⁰

Now let P be any proposition whatsoever, perhaps including occurrences of the possibility operator \diamond . In one direction, suppose that $P \in \Phi$. Then, $\Phi \vdash_{\mathcal{S}} P$, hence, by Lemma 1.7, $\Phi \vdash_{\mathcal{S}} \delta(P)$, so by maximal consistency, $\delta(P) \in \Phi$. Therefore, by the preceding result, $\Phi \models_{\rho_S} \delta(P)$. Thus, by Corollary 1.13, $\Phi \models_{\rho_S} P$. Conversely, suppose that $\Phi \models_{\rho_S} P$. Then, by Corollary 1.13, $\Phi \models_{\rho_S} \delta(P)$, so from the preceding result, $\delta(P) \in \Phi$. Therefore, from Lemma 1.7, $\Phi \vdash_{\mathcal{S}} P$, and now $P \in \Phi$ follows from Lemma 1.27. \blacksquare

We are now ready to prove that \mathcal{K} is complete.

Corollary 1.33 (Completeness of \mathcal{K}) *\mathcal{K} is complete for the class of all frames. That is, if P is valid in a frame \mathcal{F} , then $\vdash_{\mathcal{K}} P$.*

Proof: We prove the contrapositive: if $\not\vdash_{\mathcal{K}} P$ then P is not valid in every frame. Specifically, we suppose that $\not\vdash_{\mathcal{K}} P$ and will proceed to show that P is not valid in the canonical frame for \mathcal{K} , $\mathcal{F}_{\mathcal{K}}$. Observe first that $\{\neg P\}$ is \mathcal{K} -consistent, for otherwise Lemma 1.26 would imply $\emptyset \vdash_{\mathcal{K}} P$, contradicting our supposition $\not\vdash_{\mathcal{K}} P$. But if $\{\neg P\}$ is \mathcal{K} -consistent then Lemma 1.29 entails that there is a maximally \mathcal{K} -consistent set Φ such that $\neg P \in \Phi$. Therefore, by Theorem 1.32,

$$\Phi \models_{\rho_{\mathcal{K}}} \neg P$$

i.e., $\Phi \not\models_{\rho_{\mathcal{K}}} P$ (where $\rho_{\mathcal{K}}$ is the canonical \mathcal{K} -interpretation), and this shows that P is not valid in the frame $\mathcal{F}_{\mathcal{K}}$. \blacksquare

¹⁰Note that the reason why the induction went through was because if P is a proposition that contains no occurrences of \diamond , then no subproposition of P contains any such occurrences either.

In what follows we will often need to compare the derivability relations $\vdash_{\mathcal{S}}$ and $\vdash_{\mathcal{S}'}$ of two standard modal DPLs \mathcal{S} and \mathcal{S}' . To make such comparisons precise, we introduce the following notions: we say that \mathcal{S}' *extends* \mathcal{S} (or that \mathcal{S}' “is an extension of” \mathcal{S}), written $\mathcal{S}' \succeq \mathcal{S}$, iff $\Phi \vdash_{\mathcal{S}'} P$ whenever $\Phi \vdash_{\mathcal{S}} P$. We then define $\mathcal{S}' \succ \mathcal{S}$ to hold iff $\mathcal{S}' \succeq \mathcal{S}$ but $\mathcal{S} \not\succeq \mathcal{S}'$. When $\mathcal{S}' \succ \mathcal{S}$ we will call \mathcal{S}' a *proper extension* of \mathcal{S} . Thus, by definition, we have $\mathcal{S} \succeq \mathcal{K}$ for every standard modal DPL \mathcal{S} . Note, however, that there is nothing in the definition of a standard modal DPL to warrant that \mathcal{S} will be a *proper* extension of \mathcal{K} , even if \mathcal{S} has additional rules that \mathcal{K} does not have; for those rules might turn out to contribute nothing to the derivability relation $\vdash_{\mathcal{K}}$. The reader will verify the following facts:

Lemma 1.34 \succeq is a quasi-order, i.e., reflexive and transitive. \succ is a strict partial order, i.e., irreflexive, asymmetric, and transitive.

1.4 Some standard modal DPLs

As the soundness result for it suggests, \mathcal{K} is the weakest possible modal system: the only propositions we can prove in it are those that are valid in *every* frame. That makes \mathcal{K} uncontroversial but also somewhat uninteresting: the results are too general. By imposing additional constraints on the behavior of the modal operators—and enforcing those constraints either via new axioms or new inference rules—we arrive at logics that are sound only with respect to certain classes of frames, with the latter usually having elegant characterizations on the basis of the structure of their accessibility relations. The rest of this section presents a number of DPLs for some well-known modal systems that can be obtained by strengthening \mathcal{K} .

1.4.1 \mathcal{D}

\mathcal{D} is obtained from \mathcal{K} by adding a unary primitive rule **D** with the following semantics:

$$\beta \cup \{\Box P\} \vdash \mathbf{D} \Box P \rightsquigarrow \Diamond P. \quad (1.35)$$

In other words **D** derives $\Diamond P$ from $\Box P$, and thus essentially asserts the proposition $\Box P \Rightarrow \Diamond P$: if something is necessary then it is possible.

First note that this cannot be proved in \mathcal{K} : consider a frame $\langle \{w\}, \emptyset \rangle$ consisting of a sole world w that “sees” nothing—not even itself. Pick any interpretation ρ and proposition P . By definition, we have $w \models_{\rho} \Box P$, as there is no w' that is seen by w and in which P does not hold (vacuously, since nothing is seen by w). Yet we do not have $w \models_{\rho} \Diamond P$ because, by the same token, there does not exist any w' that is seen by w and in which P *does* hold. Therefore, $w \not\models_{\rho} \Box P \Rightarrow \Diamond P$.

Now by the soundness theorem for \mathcal{K} we know that if $\Box P \Rightarrow \Diamond P$ were provable then it would have to be valid in every frame, including the frame of the previous paragraph. As this is not the case, we conclude that $\Box P \Rightarrow \Diamond P$ is not a theorem of \mathcal{K} . Thus we know that \mathcal{D} will be a proper extension of \mathcal{K} : the new rule **D** will enable us to derive propositions that were not previously derivable. We conclude:

Theorem 1.35 *There is a P such that $\vdash_{\mathcal{D}} P$ but $\not\vdash_{\mathcal{K}} P$. Accordingly, $\mathcal{D} \succ \mathcal{K}$.*

From now on let us agree to call a frame *reflexive* (or *symmetric*, *transitive*, etc.) according to whether its accessibility relation is reflexive (or symmetric, transitive, etc.). And recall that a binary relation R is called *serial* iff for every x there is some y such that $x R y$. The following is the key observation about \mathcal{D} :

Lemma 1.36 $\Box P \models^{\mathcal{F}} \Diamond P$ for all serial frames \mathcal{F} .

Proof: Suppose that $w \models_{\rho} \Box P$ (for some arbitrary world w of \mathcal{F} and interpretation $\rho \in \mathcal{A}_{\mathcal{F}}$), so that $u \models_{\rho} P$ for every world u such that $w R_{\mathcal{F}} u$. Since there is in fact at least one world u such that $w R_{\mathcal{F}} u$ (as $R_{\mathcal{F}}$ is serial), we conclude that $u \models_{\rho} P$, and hence that $w \models_{\rho} \Diamond P$. ■

Corollary 1.37 (Soundness of \mathcal{D}) *If $\vdash_{\mathcal{D}} P$ then P is valid in every serial frame.*

Proof: The result will follow automatically from the stronger assertion that for every serial frame \mathcal{F} , $\beta \models^{\mathcal{F}} P$ whenever $\vdash_{\mathcal{D}} \beta \vdash D \rightsquigarrow P$, which is the analogue of Lemma 1.11 for \mathcal{D} . The proof of this stronger claim proceeds exactly as the proof of Lemma 1.11, which is possible because every logical implication $\models^{\mathcal{F}}$ that was demonstrated there was shown to hold for any frame \mathcal{F} whatsoever, hence including serial frames. The only place that requires additional argument is the point where we consider primitive inference rules, since we now have to account for **D**. But in that case the result is immediate from the preceding lemma. ■

Lemma 1.38 *The canonical \mathcal{D} -frame is serial.*

Proof: Let Φ be any world in $\mathcal{F}_{\mathcal{D}}$, and pick any proposition of the form $\Box Q \in \Phi$ (such a proposition must exist by 1.31). Then, by virtue of **D**, we have $\Phi \vdash_{\mathcal{D}} \Diamond Q$, hence, by Lemma 1.27, $\Diamond Q \in \Phi$. Accordingly, by Lemma 1.30, the set

$$\{P \mid \Box P \in \Phi\} \tag{1.36}$$

is \mathcal{D} -consistent. Hence, Lemma 1.29 implies that there is a maximally \mathcal{D} -consistent set Ψ that contains set 1.36. Moreover, by construction, we have $P \in \Psi$ for every $\Box P \in \Psi$. Therefore, by the definition of $R_{\mathcal{D}}$, we get $\Phi R_{\mathcal{D}} \Psi$, and we have thus shown that for every $\Phi \in W_{\mathcal{D}}$ there is a $\Psi \in W_{\mathcal{D}}$ such that $\Phi R_{\mathcal{D}} \Psi$; i.e., that $R_{\mathcal{D}}$ is serial. ■

Corollary 1.39 (Completeness of \mathcal{D}) *\mathcal{T} is complete for the class of serial frames. That is, if P is valid in a serial frame then $\vdash_{\mathcal{D}} P$.*

Proof: The proof proceeds just like the proof of Corollary 1.33, except that now we are dealing with \mathcal{D} and the class of serial frames, rather than \mathcal{K} and the class of all frames. Accordingly, we show that if $\not\vdash_{\mathcal{D}} P$ then P is not valid in every serial frame, and in particular, P is not valid in the canonical \mathcal{D} -frame, which was shown to be serial by Lemma 1.38. On the assumption $\not\vdash_{\mathcal{D}} P$, $\{\neg P\}$ must be \mathcal{D} -consistent, for otherwise we would have $\emptyset \vdash_{\mathcal{D}} P$ by Lemma 1.26, contradicting the assumption $\not\vdash_{\mathcal{D}} P$. But if $\{\neg P\}$ is \mathcal{D} -consistent then, by Lemma 1.29, there must exist a maximally \mathcal{D} -consistent set $\Phi \supseteq \{\neg P\}$, so by Theorem 1.32, $\Phi \models_{\rho_{\mathcal{D}}} \neg P$, where $\rho_{\mathcal{D}}$ is the canonical \mathcal{D} -interpretation. This shows that P is not valid in the canonical \mathcal{D} -frame. ■

1.4.2 \mathcal{T}

\mathcal{T} is also obtained from \mathcal{K} , through the addition of one extra primitive inference rule, **T**, whose semantics are specified by the following evaluation axiom:

$$\beta \cup \{\Box P\} \vdash \mathbf{T} \Box P \rightsquigarrow P. \quad (1.37)$$

In other words **T** derives P from $\Box P$, and thus essentially asserts the proposition $\Box P \Rightarrow P$: if something is necessary, then it holds.

Again it should first be observed that this cannot be proved in \mathcal{K} : consider a frame $\langle \{w_1, w_2\}, \{\langle w_1, w_2 \rangle\} \rangle$ (where w_1 can “see” w_2), and an interpretation ρ that makes an atom A true in w_2 and false in w_1 . Then $\Box A$ holds in w_1 (with respect to ρ), since it holds in every world that w_1 can see (namely, w_2); yet A does not hold in w_1 itself, by our specification of ρ . In symbols, $w_1 \models_{\rho} \Box A$ while $w_1 \not\models_{\rho} A$, and hence it follows from the semantics of \Rightarrow that $w_1 \not\models_{\rho} \Box A \Rightarrow A$.

Now the underderivability of $\Box P \Rightarrow P$ in \mathcal{K} can be shown by the same line of reasoning that we used to show the underderivability of $\Box P \Rightarrow \Diamond P$ in \mathcal{K} : if, by way of contradiction, $\Box P \Rightarrow P$ were provable in \mathcal{K} , then it would have to be valid in every frame (by Corollary 1.12). But as we just showed in the last paragraph, $\Box P \Rightarrow P$ is *not* valid in every frame. Therefore, $\not\vdash_{\mathcal{K}} \Box P \Rightarrow P$.

Theorem 1.40 *There is a P such that $\vdash_{\mathcal{T}} P$ and $\not\vdash_{\mathcal{K}} P$. Therefore, $\mathcal{T} \succ \mathcal{K}$.*

Next we consider the relationship between \mathcal{T} and \mathcal{D} . Intuitively, the proposition $\Box P \Rightarrow P$ makes a bolder assertion than the “ \mathcal{D} -axiom” $\Box P \Rightarrow \Diamond P$. All the latter says is that if P is necessary then it is possible—not that it is actually true, only that it is possible. Thus \mathcal{T} appears to be stronger than \mathcal{D} , and indeed we can prove that this is the case by showing that the formula $\Box P \Rightarrow \Diamond P$ is derivable in \mathcal{T} :

```

assume  $\Box P$  in
  begin
    suppose-absurd  $\Box \neg P$  in
      begin
        T  $\Box \neg P$ ;
        T  $\Box P$ ;
        absurd  $P, \neg P$ 
      end;
     $\diamond$ -intro  $\neg \Box \neg P$ 
  end

```

Thus every application of rule **D** to a proposition $\Box P$, in any given deduction of \mathcal{D} , can be replaced by the body of the foregoing hypothetical deduction, and this effectively desugars every deduction of \mathcal{D} into an observationally equivalent deduction of \mathcal{T} , which shows that **D** does not at all increase the power of \mathcal{T} .¹¹

We conclude:

Theorem 1.41 $\mathcal{T} \succeq \mathcal{D}$. *That is, if $\Phi \vdash_{\mathcal{D}} P$ then $\Phi \vdash_{\mathcal{T}} P$.*

The question that is raised now is whether there is a desugaring in the reverse direction, allowing us to convert any deduction of \mathcal{T} into an equivalent deduction of \mathcal{D} . If so, \mathcal{T} and \mathcal{D} would be stylistic variations of each other: both would lead to the exact same set of theorems. The answer is straightforward once we realize that the “ \mathcal{T} -axiom” $\Box P \Rightarrow P$ is not provable in \mathcal{D} . By the soundness theorem for \mathcal{D} , to show that $\Box P \Rightarrow P$ is not provable in it we only need to produce a serial frame in which $\Box P \Rightarrow P$ is not valid. On the assumption that such a frame \mathcal{F} exists, $\Box P \Rightarrow P$ cannot be a theorem of \mathcal{D} , because if it were then it would have to be valid in every serial

¹¹More formally, we should define a mapping σ from deductions of \mathcal{D} to deductions of \mathcal{T} as follows:

```

 $\sigma(P) = P$ 
 $\sigma(\mathbf{D} \Box P) = \text{suppose-absurd } \Box \neg P \text{ in}$ 
  begin
    T  $\Box \neg P$ ; T  $\Box P$ ; absurd  $P, \neg P$ 
  end;
 $\diamond$ -intro  $\neg \Box \neg P$ 
 $\sigma(\text{Prim-Rule } P_1, \dots, P_n) = \text{Prim-Rule } P_1, \dots, P_n$  (for  $\text{Prim-Rule} \neq \mathbf{D}$ )
 $\sigma(\text{assume } P \text{ in } D) = \text{assume } P \text{ in } \sigma(D)$ 
 $\sigma(D_1; D_2) = \sigma(D_1); \sigma(D_2)$ 
 $\sigma(\text{nec: } D) = \text{nec: } \sigma(D)$ 

```

It is then straightforward to prove by induction on D that $\vdash_{\mathcal{T}} \beta \vdash \sigma(D) \rightsquigarrow P$ whenever $\vdash_{\mathcal{D}} \beta \vdash D \rightsquigarrow P$.

frame, including \mathcal{F} . Now such a frame \mathcal{F} is easy to produce: let $W_{\mathcal{F}} = \{w_1, w_2\}$ and $R_{\mathcal{F}} = \{\langle w_1, w_2 \rangle, \langle w_2, w_1 \rangle\}$, so that there are two worlds that can see each other (but not themselves), and let ρ be an interpretation that makes an atom A true in w_2 but false in w_1 . Then $w_1 \models_{\rho} \Box A$ but $w_1 \not\models_{\rho} A$, hence $w_1 \not\models_{\rho} \Box A \Rightarrow A$. We conclude:

Theorem 1.42 *There is a P such that $\vdash_{\mathcal{T}} P$ but $\not\models_{\mathcal{D}} P$. Therefore, by Theorem 1.41, $\mathcal{T} \succ \mathcal{D}$.*

Note that Theorem 1.40 could now be alternatively derived directly from Theorem 1.42, Theorem 1.35, and the transitivity of \succeq .

Another proposition that we can derive in \mathcal{T} is $P \Rightarrow \Diamond P$. This fact will prove useful later on, so we state and prove it here:

Lemma 1.43 $\vdash_{\mathcal{T}} P \Rightarrow \Diamond P$.

Proof: Let D be the following:

```

assume  $P$  in
  begin
    assume  $\Box \neg P$  in
      T  $\Box \neg P$ ;
    suppose-absurd  $\neg P$  in
      absurd  $P, \neg P$ ;
    modus-tollens  $\Box \neg P \Rightarrow \neg P, \neg \neg P$ ;
     $\Diamond$ -intro  $\neg \Box \neg P$ 
  end

```

It is readily verified that $\vdash_{\mathcal{T}} \emptyset \vdash D \rightsquigarrow P \Rightarrow \Diamond P$. ■

Finally, we prove the soundness of \mathcal{T} , which centers on the following observation:

Lemma 1.44 $\Box P \models^{\mathcal{F}} P$ for all reflexive frames \mathcal{F} .

Proof: Suppose that $w \models_{\rho} \Box P$ (for some arbitrary world w of \mathcal{F} and interpretation $\rho \in \mathcal{A}_{\mathcal{F}}$), so that $u \models_{\rho} P$ for every world u such that $w R_{\mathcal{F}} u$. But w is itself such a world since, by reflexivity, we have $w R_{\mathcal{F}} w$. Hence $w \models_{\rho} P$. ■

Corollary 1.45 (Soundness of \mathcal{T}) *If $\vdash_{\mathcal{T}} P$ then P is valid in every reflexive frame.*

Proof: The proof is similar to that of Corollary 1.37. The result follows immediately from the stronger assertion that for every reflexive frame \mathcal{F} , $\beta \models^{\mathcal{F}} P$ whenever $\vdash_{\mathcal{T}} \beta \vdash D \rightsquigarrow P$, which is the analogue of Lemma 1.11 for \mathcal{T} . The proof of this stronger claim proceeds exactly as the proof of Lemma 1.11, the only difference being in the treatment of primitive rules, since we now have to account for **T**. But in that case the result is immediate from the preceding lemma. ■

Lemma 1.46 *The canonical \mathcal{T} -frame is reflexive.*

Proof: Let Φ be any world in $W_{\mathcal{T}}$. By definition, we have $\Phi R_{\mathcal{T}} \Phi$ iff $P \in \Phi$ whenever $\Box P \in \Phi$. Now suppose $\Box P \in \Phi$. Then $\Phi \vdash_{\mathcal{T}} P$ (by virtue of \mathcal{T}), hence, since Φ is deductively closed, $P \in \Phi$. This shows that $\Phi R_{\mathcal{T}} \Phi$ for all $\Phi \in W_{\mathcal{T}}$, and hence that $R_{\mathcal{T}}$ is reflexive. ■

Corollary 1.47 (Completeness of \mathcal{T}) *\mathcal{T} is complete for the class of reflexive frames. That is, if P is valid in a reflexive frame then $\vdash_{\mathcal{T}} P$.*

Proof: Identical to the proof of Corollary 1.39, using \mathcal{T} instead of \mathcal{D} , reflexive instead of serial frames, and Lemma 1.46 in place of Lemma 1.38. ■

1.4.3 \mathcal{S}_4

\mathcal{S}_4 is obtained from \mathcal{T} by adding one more primitive inference rule, \mathbf{S}_4 , with the following semantics:

$$\beta \cup \{\Box P\} \vdash_{\mathbf{S}_4} \Box P \rightsquigarrow \Box \Box P. \quad (1.38)$$

So \mathbf{S}_4 derives $\Box \Box P$ from $\Box P$, and thus essentially asserts the following conditional:

$$\Box P \Rightarrow \Box \Box P. \quad (1.39)$$

That is, if something is necessary then it is so not by accident but by necessity—necessary truths are *necessarily* necessary.

Philosophically, this is a debatable proposition, but we will not be concerned much about that here. Let us instead work out the proof-theoretic relationship of this DPL to the ones we have already introduced, and see what kind of soundness result we might obtain for it.

We begin by showing that the new rule cannot be desugared in \mathcal{T} (and hence neither in \mathcal{K} nor in \mathcal{D} , since \mathcal{T} is the strongest of the three). This can be achieved if we can show that proposition 1.39 is not provable in \mathcal{T} . As should be familiar by now, we will demonstrate this with reasoning by contradiction via the soundness theorem for \mathcal{T} : we will produce a reflexive frame \mathcal{F} in which 1.39 is not valid; this will establish that 1.39 is not provable in \mathcal{T} , for if it were then it would have to be valid in every reflexive frame.

Consider a reflexive frame \mathcal{F} consisting of three worlds w_1 , w_2 , and w_3 , where w_1 can see w_2 and w_2 can see w_3 , but w_2 cannot see w_1 and w_3 can see neither w_1 nor w_2 . That is,

$$\mathcal{F} = \langle \{w_1, w_2, w_3\}, \{ \langle w_1, w_1 \rangle, \langle w_1, w_2 \rangle, \langle w_2, w_2 \rangle, \langle w_2, w_3 \rangle, \langle w_3, w_3 \rangle \} \rangle.$$

Further, consider an interpretation that makes an atom A true in w_1 and w_2 , but false in w_3 . Then $\Box A$ holds in w_1 , but $\Box\Box A$ does not: for $\Box\Box A$ to hold in w_1 , the proposition $\Box A$ must be true in every world that w_1 sees, which includes w_2 . But $\Box A$ is not true in w_2 , since w_2 sees w_3 and A is not true in w_3 . This shows that $\Box P \Rightarrow \Box\Box P$ is not valid in every reflexive frame, and hence that it is not a theorem of \mathcal{T} . Accordingly, \mathcal{S}_4 is a proper extension of \mathcal{T} : some theorems of \mathcal{S}_4 (such as 1.39) are not provable in \mathcal{T} .

Theorem 1.48 $\mathcal{S}_4 \succ \mathcal{T}$.

Regarding soundness, the basic intuition is that by iterating the modal operator \Box (producing $\Box\Box P$ from $\Box P$), rule \mathbf{S}_4 expresses a sort of transitivity. This is borne out by the following result:

Lemma 1.49 $\Box P \models^{\mathcal{F}} \Box\Box P$ for all transitive frames \mathcal{F} .

Proof: Pick any \mathcal{F} -world w and $\rho \in \mathcal{A}_{\mathcal{F}}$, for \mathcal{F} transitive, and suppose that

$$w \models_{\rho} \Box P \tag{1.40}$$

so that

$$u \models_{\rho} P \tag{1.41}$$

for every u such that

$$w R_{\mathcal{F}} u. \tag{1.42}$$

Pick any such u , and consider any v such that

$$u R_{\mathcal{F}} v. \tag{1.43}$$

Since $R_{\mathcal{F}}$ is transitive, 1.42 and 1.43 imply $w R_{\mathcal{F}} v$, and hence, by 1.40, we must have $v \models_{\rho} P$. This shows that $u \models_{\rho} \Box P$, and since u was chosen as an arbitrary world seen by w , it follows that $w \models_{\rho} \Box\Box P$. \blacksquare

Corollary 1.50 (Soundness of \mathcal{S}_4) *If $\vdash_{\mathcal{S}_4} P$ then P is valid in every reflexive and transitive frame.*

Proof: Identical to the proof of Corollary 1.45, with the exception of the new rule \mathbf{S}_4 , which is directly handled by the previous lemma. \blacksquare

Lemma 1.51 *The canonical \mathcal{S}_4 -frame is reflexive and transitive.*

Proof: Showing that $\Phi R_{\mathcal{S}_4} \Phi$ for every $\Phi \in W_{\mathcal{S}_4}$ is achieved just as in \mathcal{T} , since \mathcal{S}_4 contains \mathcal{T} . For transitivity, suppose we have

$$\Phi_1 R_{\mathcal{S}_4} \Phi_2 \tag{1.44}$$

and

$$\Phi_2 R_{\mathcal{S}_4} \Phi_3 \tag{1.45}$$

for arbitrary worlds Φ_1 , Φ_2 , and Φ_3 in $W_{\mathcal{S}_4}$. To show $\Phi_1 R_{\mathcal{S}_4} \Phi_3$, consider any proposition of the form $\Box P \in \Phi_1$. By virtue of \mathbf{S}_4 , we have $\Phi_1 \vdash_{\mathcal{S}_4} \Box \Box P$, and since Φ_1 is maximally \mathcal{S}_4 -consistent, Lemma 1.27 yields

$$\Box \Box P \in \Phi_1. \tag{1.46}$$

Now from 1.44, 1.46, and the definition of $R_{\mathcal{S}_4}$ we get

$$\Box P \in \Phi_2. \tag{1.47}$$

while from 1.47 and 1.45 we get $P \in \Phi_3$. We have thus shown that $P \in \Phi_3$ whenever $\Box P \in \Phi_1$, i.e., that $\Phi_1 R_{\mathcal{S}_4} \Phi_3$. ■

Corollary 1.52 (Completeness of \mathcal{S}_4) *\mathcal{S}_4 is complete for the class of reflexive and transitive frames. That is, if P is valid in a reflexive and transitive frame, then $\vdash_{\mathcal{S}_4} P$.*

Proof: Identical to the proof of Corollary 1.39, but using \mathcal{S}_4 instead of \mathcal{D} , reflexive and transitive instead of serial frames, and Lemma 1.51 in place of Lemma 1.38. ■

1.4.4 \mathcal{S}_5

The last modal DPL we will introduce in this paper is \mathcal{S}_5 , which is obtained from \mathcal{T} via the addition of a new primitive rule \mathbf{S}_5 , with the following semantics:

$$\beta \cup \{\Diamond P\} \vdash_{\mathbf{S}_5} \Diamond P \rightsquigarrow \Box \Diamond P. \tag{1.48}$$

By deriving $\Box \Diamond P$ from $\Diamond P$, \mathbf{S}_5 essentially asserts the proposition $\Diamond P \Rightarrow \Box \Diamond P$: if something is possible then it is necessarily possible.

We begin by showing that \mathcal{S}_5 is a proper extension of \mathcal{S}_4 , and hence of \mathcal{T} as well.

Lemma 1.53 $\vdash_{\mathcal{S}_5} \Diamond \Box P \Rightarrow \Box P$.

Proof: We have $\Box\Diamond\neg P \equiv \neg\Diamond\Box P$, therefore, by Lemma 1.6 and the fact that $\mathcal{S}_5 \succeq \mathcal{K}$, there is a D_1 such that

$$\vdash_{\mathcal{S}_5} \emptyset \vdash D_1 \rightsquigarrow \Box\Diamond\neg P \Leftrightarrow \neg\Diamond\Box P.$$

Likewise, $\neg\Diamond\neg P \equiv \Box P$, so there is a D_2 such that $\vdash_{\mathcal{S}_5} \emptyset \vdash D_2 \rightsquigarrow \neg\Diamond\neg P \Leftrightarrow \Box P$. Let D be the following:

```

D1;
D2;
assume  $\Diamond\Box P$  in
  begin
    suppose-absurd  $\Diamond\neg P$  in
      begin
        S5  $\Diamond\neg P$ ;
        left-iff  $\Box\Diamond\neg P \Leftrightarrow \neg\Diamond\Box P$ ;
        absurd  $\Diamond\Box P, \neg\Diamond\Box P$ 
      end;
    left-iff  $\neg\Diamond\neg P \Leftrightarrow \Box P$ ;
    modus-ponens  $\neg\Diamond\neg P \Rightarrow \Box P, \neg\Diamond\neg P$ 
  end
end

```

We now have $\vdash_{\mathcal{S}_5} \emptyset \vdash D \rightsquigarrow \Diamond\Box P \Rightarrow \Box P$. ■

Lemma 1.54 $\vdash_{\mathcal{S}_5} \Box P \Rightarrow \Box\Box P$.

Proof: Since $\mathcal{S}_5 \succeq \mathcal{T}$, Lemma 1.43 implies that there is a D_1 such that

$$\vdash_{\mathcal{S}_5} \emptyset \vdash D_1 \rightsquigarrow \Box P \Rightarrow \Diamond\Box P.$$

Further, from Lemma 1.43, Lemma 1.53, and part (e) of Lemma 1.19 it follows that $\vdash_{\mathcal{S}_5} \Diamond\Box P \Leftrightarrow \Box P$, so by Lemma 1.4 we conclude that there is a D_2 such that

$$\vdash_{\mathcal{S}_5} \emptyset \vdash D_2 \rightsquigarrow \Box\Diamond\Box P \Leftrightarrow \Box\Box P.$$

Now let D be the following deduction:

```

D1;
D2;
assume  $\Box P$  in
  begin

```

modus-ponens $\Box P \Rightarrow \Diamond \Box P, \Box P$;
 \mathbf{S}_5 $\Diamond \Box P$;
left-iff $\Box \Diamond \Box P \Leftrightarrow \Box \Box P$;
modus-ponens $\Box \Diamond \Box P \Rightarrow \Box \Box P, \Box \Diamond \Box P$
end

The reader will verify that $\vdash_{\mathcal{S}_5} \emptyset \vdash D \rightsquigarrow \Box P \Rightarrow \Box \Box P$. ■

It now follows that \mathcal{S}_5 is an extension of \mathcal{S}_4 , as every application of \mathbf{S}_4 to a proposition of the form $\Box P$ can be replaced by the \mathcal{S}_5 -deduction

$$D; \text{modus-ponens } \Box P \Rightarrow \Box \Box P, \Box P$$

where D is the deduction defined in the proof of Lemma 1.54. Lemma 1.55 below will moreover show \mathcal{S}_5 to be a proper extension of \mathcal{S}_4 (and hence of \mathcal{T} as well):

Lemma 1.55 *There is a reflexive and transitive frame in which $\Diamond P \Rightarrow \Box \Diamond P$ is not valid.*

Proof: Let \mathcal{F} consist of two worlds w_1 and w_2 that can see themselves (so that \mathcal{F} is reflexive), and where w_1 can see w_2 but not conversely. It is readily verified that this frame is also transitive. Now let ρ be an interpretation that makes an atom A true in w_1 and false in w_2 . Then we have $w_1 \models_\rho \Diamond A$, since w_1 sees a world (namely itself) where A is true. However, we do not have $w_1 \models_\rho \Box \Diamond A$, because $\Diamond A$ does not hold in *every* world that w_1 sees—a counterexample is w_2 , where $\Diamond A$ does not hold owing to the fact that A fails in every world visible to w_2 (of course there is only such world, namely w_2 itself). Therefore, $w_1 \models_\rho \Diamond A$ but $w_1 \not\models_\rho \Box \Diamond A$, which is to say $w_1 \not\models_\rho \Diamond A \Rightarrow \Box \Diamond A$. ■

It now follows from the soundness theorem of \mathcal{S}_4 that the proposition $\Diamond P \Rightarrow \Box \Diamond P$ is derivable in neither of these two DPLs. We conclude:

Theorem 1.56 $\mathcal{S}_5 \succ \mathcal{S}_4 \succ \mathcal{T} \succ \mathcal{D} \succ \mathcal{K}$.

The frame that provided the countermodel in the proof of Lemma 1.55 was reflexive and transitive but not symmetric. Would the addition of symmetry rule out countermodels for the “ \mathcal{S}_5 -axiom” $\Diamond P \Rightarrow \Box \Diamond P$? The following result answers that affirmatively:

Lemma 1.57 $\Diamond P \models^{\mathcal{F}} \Box \Diamond P$ for all symmetric and transitive frames \mathcal{F} .

Proof: Let \mathcal{F} be a symmetric and transitive frame. Pick any \mathcal{F} -world w and $\rho \in \mathcal{A}_{\mathcal{F}}$, and suppose that

$$w \models_{\rho} \diamond P \quad (1.49)$$

so that

$$u \models_{\rho} P \quad (1.50)$$

for some u such that

$$w R_{\mathcal{F}} u. \quad (1.51)$$

By symmetry, $v R_{\mathcal{F}} w$, so by 1.51 and transitivity we get $v R_{\mathcal{F}} u$. Accordingly, 1.50 yields $v \models_{\rho} \diamond P$. Since this holds for any v that is seen by w , we conclude that $w \models_{\rho} \Box \diamond P$. ■

This allows us to prove the following soundness result for \mathcal{S}_5 :

Corollary 1.58 (Soundness of \mathcal{S}_5) *If $\vdash_{\mathcal{S}_5} P$ then P is valid in every reflexive, symmetric, and transitive frame. That is, a theorem of \mathcal{S}_5 is valid in every frame whose accessibility relation is an equivalence.*

Proof: The argument proceeds exactly as in the proof of Corollary 1.45, with the case of the new rule \mathbf{S}_5 covered by the preceding lemma. ■

For completeness, we need one more provability fact about \mathcal{S}_5 :

Lemma 1.59 $\vdash_{\mathcal{S}_5} \neg P \Rightarrow \Box \neg \Box P$.

Proof: Since $\mathcal{S}_5 \succeq \mathcal{T}$, Lemma 1.43 implies that there is a D_1 such that

$$\vdash_{\mathcal{S}_5} \emptyset \vdash D_1 \rightsquigarrow \neg P \Rightarrow \diamond \neg P.$$

Further, $\Box \diamond \neg P \equiv \Box \neg \Box P$, therefore, owing to Lemma 1.6 and the inclusion $\mathcal{S}_5 \succeq \mathcal{K}$, there is a D_2 such that

$$\vdash_{\mathcal{S}_5} \emptyset \vdash D_2 \rightsquigarrow \Box \diamond \neg P \Leftrightarrow \Box \neg \Box P.$$

Now let D be the following deduction:

D_1 ;

D_2 ;

assume $\neg P$ in

begin

modus-ponens $\neg P \Rightarrow \diamond \neg P, \neg P$;

\mathbf{S}_5 $\diamond \neg P$;

left-iff $\Box \diamond \neg P \Leftrightarrow \Box \neg \Box P$;

modus-ponens $\Box \diamond \neg P \Rightarrow \Box \neg \Box P, \Box \diamond \neg P$

end

It is straightforward to verify that $\vdash_{\mathcal{S}_5} \emptyset \vdash D \rightsquigarrow \neg P \Rightarrow \Box \neg \Box P$. ■

We can now prove:

Lemma 1.60 *The canonical \mathcal{S}_5 -frame is reflexive, symmetric, and transitive.*

Proof: Reflexivity and transitivity are shown just as in \mathcal{T} and \mathcal{S}_4 , respectively, since \mathcal{S}_5 is an extension of both. For symmetry, suppose that

$$\Phi_1 R_{\mathcal{S}_5} \Phi_2 \tag{1.52}$$

for any $\Phi_1, \Phi_2 \in W_{\mathcal{S}_5}$. To prove $\Phi_2 R_{\mathcal{S}_5} \Phi_1$, we need to show that if $\Box P \in \Phi_2$ then $P \in \Phi_1$, for any P ; or equivalently, that if $P \notin \Phi_1$ then $\Box P \notin \Phi_2$. Accordingly, suppose that $P \notin \Phi_1$. Then, by part (a) of Lemma 1.28, $\neg P \in \Phi_1$, hence

$$\Phi_1 \vdash_{\mathcal{S}_5} \neg P \tag{1.53}$$

while, from Lemma 1.59,

$$\Phi_1 \vdash_{\mathcal{S}_5} \neg P \Rightarrow \Box \neg \Box P. \tag{1.54}$$

Now from 1.53, 1.54, and part (d) of Lemma 1.19 we infer $\Phi_1 \vdash_{\mathcal{S}_5} \Box \neg \Box P$, and so, by Lemma 1.27, $\Box \neg \Box P \in \Phi_1$. Hence, by 1.52,

$$\neg \Box P \in \Phi_2 \tag{1.55}$$

and now $\Box P \notin \Phi_2$ follows from 1.55 and part (a) of Lemma 1.28. ■

Corollary 1.61 (Completeness of \mathcal{S}_5) *\mathcal{S}_5 is complete for the class of reflexive, symmetric, and transitive frames. That is, if P is valid in a reflexive, symmetric, and transitive frame, then $\vdash_{\mathcal{S}_5} P$.*

Proof: Identical to the proof of Corollary 1.52, but using \mathcal{S}_5 instead of \mathcal{S}_4 , reflexive, symmetric, and transitive frames instead of reflexive and transitive frames, and Lemma 1.60 instead of Lemma 1.51. ■

Bibliography

- [1] K. Arkoudas. Denotational Proof Languages. PhD thesis, MIT, 2000.
- [2] A. Chagrov and M. Zakharyashev. *Modal Logic*, volume 35 of *Oxford Logic Guides*. Oxford Science Publications, 1997.
- [3] B. F. Chellas. *Modal Logic: an Introduction*. Cambridge University Press, 1980.
- [4] M. Fitting. *Proof methods for modal and intuitionistic logics*. Reidel Publishing, Holland, 1983.
- [5] G. E. Hughes and M. J. Cresswell. *A new introduction to modal logic*. Routledge, London, England, 1968.
- [6] E. J. Lemmon and D. Scott. *The 'Lemmon Notes': an introduction to Modal Logic*. ed. K. Segerberg, Oxford, Basil Blackwell, 1977.