On the Impact of Player Capability on Congestion Games

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Abstract. We study the impact of player capability on social welfare in congestion games. We introduce a new game, the ***D****istance-bounded* ***N****etwork* ***C****ongestion game (DNC)*, as the basis of our study. DNC is a symmetric network congestion game with a bound on the number of edges each player can use. We show that DNC is PLS-complete in contrast to standard symmetric network congestion games which are in P. To model different player capabilities, we propose using programs in a Domain-Specific Language (DSL) to compactly represent player strategies. We define a player's capability as the maximum size of the programs they can use. We introduce two variants of DNC with accompanying DSLs representing the strategy spaces. We propose four *capability preference* properties to characterize the impact of player capability on social welfare at equilibrium. We then establish necessary and sufficient conditions for the four properties in the context of our DNC variants. Finally, we study a specific game where we derive exact expressions of the social welfare in terms of the capability bound. This provides examples where the social welfare at equilibrium increases, stays the same, or decreases as players become more capable.

Keywords: Congestion game · Player capability · Social welfare.

1 Introduction

Varying player capabilities can significantly affect the outcomes of strategic games. Developing a comprehensive understanding of how different player capabilities affect the dynamics and overall outcomes of strategic games is therefore an important long-term research goal in the field. Central questions include characterizing, and ideally precisely quantifying, player capabilities, then characterizing, and ideally precisely quantifying, how these different player capabilities interact with different game characteristics to influence or even fully determine individual and/or group dynamics and outcomes.

We anticipate a range of mechanisms for characterizing player capabilities, from simple numerical parameters through to complex specifications of available player behavior. Here we use programs in a domain-specific language (DSL) to

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compactly represent player strategies. Bounding the sizes of the programs available to the players creates a natural capability hierarchy, with more capable players able to deploy more diverse strategies defined by larger programs. Building on this foundation, we study the effect of increasing or decreasing player capabilities on game outcomes such as social welfare at equilibrium. To the best of our knowledge, this paper presents the first systematic analysis of the effect of different player capabilities on the outcomes of strategic games.

We focus on network congestion games [8, 23]. All congestion games have pure Nash equilibria (PNEs) [19, 22]. Congestion games have been applied in many areas including drug design [18], load balancing [26], and network design [14]. There is a rich literature on different aspects of congestion games including their computational characteristics [1], efficiency of equilibria [7], and variants such as weighted congestion games [6] or games with unreliable resources [17].

We propose a new network congestion game, the ***D**istance-bounded **N**etwork **C**ongestion game (DNC)*, as the basis of our study. A network congestion game consists of some players and a directed graph where each edge is associated with a delay function. The goal of each player is to plan a path that minimizes the delay from a source vertex to a sink vertex. The delay of a path is the sum of the delays on the edges in the path, with the delay on each edge depending (only) on the number of players choosing the edge. The game is symmetric when all players share the same source and sink. Fabrikant et al. [8] shows that finding a PNE is in P for symmetric games but PLS-complete for asymmetric ones. DNC is a symmetric network congestion game in which each player is subject to a *distance bound* — i.e., a bound on the number of edges that a player can use.

We establish hardness results for DNC. We show that with the newly introduced distance bound, our symmetric network congestion game becomes PLS-complete. We also show that computing the best or worst social welfare among PNEs of DNC is NP-hard.

We then present two games for which we define compact DSLs for player strategies. The first game is a DNC variant with ***D**efault **A**ction (DncDa)*. In this game, each node has a default outgoing edge that does not count towards the distance bound. Hence a strategy can be compactly represented by specifying only the non-default choices. We establish that DncDa is as hard as DNC. The other game is Gold and Mines Game (GMG), where there are gold and mine sites placed on parallel horizontal lines, and a player uses a program to compactly describe the line that they choose at each horizontal location. We show that GMG is a special form of DncDa.

We propose four *capability preference* properties that characterize the impact of player capability on social welfare. In this paper, we only consider social welfare of pure Nash equilibria. We call a game *capability-positive* (resp. *capability-negative*) if social welfare does not decrease (resp. increase) when players become more capable. A game is *max-capability-preferred* (resp. *min-capability-preferred*) if the worst social welfare when players have maximal (resp. minimal) capability is at least as good as any social welfare when players have less (resp. more) capability. Note that max-capability-preferred (resp. min-capability-preferred) is a weaker property than capability-positive (resp. capability-negative). Due to the hardness of DncDa, we analyze a restricted version (DncDaS) where all edges share the same delay function. We identify necessary and sufficient conditions on the delay function for each capability preference property to hold universally for all configurations in the context of DncDaS and GMG (Table 1).

Finally, we study a specific version of GMG where we derive exact expressions of the social welfare in terms of the capability bound and payoff function parameterization. We present examples where the social welfare at equilibrium increases, stays the same, or decreases as players become more capable.

Summary of contributions:

- We present a framework for quantifying varying player capabilities and studying how different player capabilities affect the dynamics and outcome of strategic games. In this framework, player strategies are represented as programs in a DSL, with player capability defined as the maximal size of available programs.
- We propose four *capability preference* properties to characterize the impact of player capability on social welfare of pure Nash equilibria: *capability-positive*, *max-capability-preferred*, *capability-negative*, and *min-capability-preferred*.
- We introduce the *distance-bounded network congestion game (DNC)*, a new symmetric network congestion game in which players can only choose paths with a bounded number of edges. We further show that DNC is PLS-complete. Moreover, computing the best or worst social welfare among equilibria of DNC is NP-hard.
- We introduce two variants of DNC, DNC with default action (DncDa) and Gold and Mines Game (GMG), with accompanying DSLs to compactly represent player strategies. We then establish necessary and sufficient conditions for the capability preference properties in DncDaS and GMG.
- We study a special version of GMG where we fully characterize the social welfare at equilibrium. This characterization provides insights into the factors that affect whether increasing player capability is beneficial or not.

Additional related work There has been research exploring the results of representing player strategies using formal computational models. Tennenholtz [25] proposes using programs to represent player strategies and analyzes program equilibrium in a finite two-player game. Fortnow [9] extends the results, representing strategies as Turing machines. Another line of research uses various kinds of automata to model a player's strategy in non-congestion games [3], such as repeated prisoner's dilemma [20, 24]. Automata are typically used to model bounded rationality [15, 21] or certain learning behavior [5, 11]. Neyman [16] presents asymptotic results on equilibrium payoff in repeated normal-form games when automaton sizes meet certain conditions. There has also been research exploring structural strategy spaces in congestion games. Ackermann and Skopalik [2] considers a player-specific network congestion game where each player has a set of forbidden edges. Chan and Jiang [4] studies computing mixed Nash equilibria in a broad class of congestion games with strategy spaces compactly described by a set of linear constraints. Unlike our research, none of the above research defines a hierarchy of player capabilities or characterizes the effect of the hierarchy on game outcomes.

2 Distance-bounded network congestion game

It is well-known that computing PNEs in symmetric network congestion games belongs to P while the asymmetric version is PLS-complete [8]. We present a symmetric network congestion game where we limit the number of edges that a player can use. We show that this restriction makes the problem harder — the new game is PLS-complete, and finding the best or worst Nash equilibrium in terms of global social welfare is NP-hard.

Definition 1. *An instance of the* **D**istance-bounded **N**etwork **C**ongestion game (DNC) *is a tuple* $G = (\mathcal{V},\, \mathcal{E},\, \mathcal{N},\, s,\, t,\, b,\, (d_e)_{e\in\mathcal{E}})$ *where:*

- $\mathcal{V}$ *is the set of vertices in the network.*
- $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ *is the set of edges in the network.*
- $\mathcal{N} = \{1,\, \dots,\, n\}$ *is the set of players.*
- $s \in \mathcal{V}$ *is the source vertex shared by all players.*
- $t \in \mathcal{V}$ *is the sink vertex shared by all players.*
- $b \in \mathbb{N}$ *is the bound of the path length.*
- $d_e : \mathbb{N} \mapsto \mathbb{R}$ *is a non-decreasing delay function on edge* e*.*

We also require that the network has no negative-delay cycles, i.e., for each cycle $\mathcal{C}$*, we require* $\sum_{e\in\mathcal{C}} \min_{i\in\mathcal{N}} d_e(i) = \sum_{e\in\mathcal{C}} d_e(1) \geq 0$*.*

We only consider pure strategies (i.e., deterministic strategies) in this paper. The strategy space of a single player contains all $s-t$ *simple paths whose length does not exceed* b*:*

$$\mathcal{L}_b \overset{\text{def}}{=} \left\{ (p_0,\, \dots,\, p_k) \;\middle|\; \begin{array}{l} p_0 = s,\, p_k = t,\, (p_i,\, p_{i+1}) \in \mathcal{E}, k \leq b, \\ p_i \neq p_j \textit{ for } i \neq j \end{array} \right\}$$

In a DNC, as in a general congestion game, a player's goal is to minimize their delay. Let $s_i \overset{\text{def}}{=} (p_{i0},\, \cdots,\, p_{ik_i}) \in \mathcal{L}_b$ denote the strategy of player i where $i \in \mathcal{N}$. A *strategy profile* $\boldsymbol{s} = (s_1, \dots, s_n) \in \mathcal{L}_b^n$ consists of strategies of all players. Let $E_i \overset{\text{def}}{=} \{(p_{ij},\, p_{i,j+1}) \mid 0 \leq j < k_i\}$ denote the corresponding set of edges on the path chosen by player i. The *load* on an edge $e \in \mathcal{E}$ is defined as the number of players that occupy this edge: $x_e \overset{\text{def}}{=} |\, \{i \mid e \in E_i\}\, |$. The delay experienced by player i is $c_i(\boldsymbol{s}) \overset{\text{def}}{=} \sum_{e\in E_i} d_e(x_e)$. A strategy profile $\boldsymbol{s}$ is a *pure Nash equilibrium (PNE)* if no player can improve their delay by unilaterally changing strategy, i.e., $\forall i \in \mathcal{N} : c_i(\boldsymbol{s}) = \min_{s'\in\mathcal{L}_b} c_i(\boldsymbol{s}_{-i},\, s')$. All players experience infinite delay if the distance bound permits no feasible solution (i.e., when $\mathcal{L}_b = \emptyset$). *Social welfare* is defined as the the negative total delay of all players where a larger welfare value means on average players experience less delay: $W(\boldsymbol{s}) \overset{\text{def}}{=} -\sum_{i\in\mathcal{N}} c_i(\boldsymbol{s})$.

We now present a few hardness results about DNC.

Lemma 1. *DNC belongs to* PLS*.*

Proof. DNC is a potential game where local minima of its potential function correspond to PNEs [22]. Clearly there are polynomial algorithms for finding a feasible solution or evaluating the potential function. We only need to show that computing the best response of some player i given the strategies of others is in P. For each $v \in \mathcal{V}$, we define $f(v, d)$ to be the minimal delay experienced by player i over all paths from s to v with length bound d. It can be recursively computed via $f(v, d) = \min_{u \in \mathcal{V}:(u, v) \in \mathcal{E}}(f(u, d-1) + d_{(u, v)}(x_{(u, v)} + 1))$ where $x_{(u, v)}$ is the load on edge (u, v) caused by other players. The best response of player i is then $f(t, b)$. If there are cycles in the solution, we can remove them without affecting the total delay because cycles must have zero delay in the best response.

Theorem 1. *DNC is* PLS*-complete.*

Proof. We have shown that DNC belongs to PLS. Now we present a PLS-reduction from a PLS-complete game to finish the proof.

The *quadratic threshold game* [1] is a PLS-complete game in which there are n players and $n(n+1)/2$ resources. The resources are divided into two sets $\mathcal{R}^{\text{in}} = \{r_{ij} \mid 1 \le i < j \le n\}$ for all unordered pairs of players $\{i, j\}$ and $\mathcal{R}^{\text{out}} = \{r_i \mid i \in \mathcal{N}\}$. For ease of exposition, we use r_{ij} and r_{ji} to denote the same resource. Player i has two strategies: $S_i^{\text{in}} = \{r_{ij} \mid j \in \mathcal{N}/\{i\}\}$ and $S_i^{\text{out}} = \{r_i\}$.

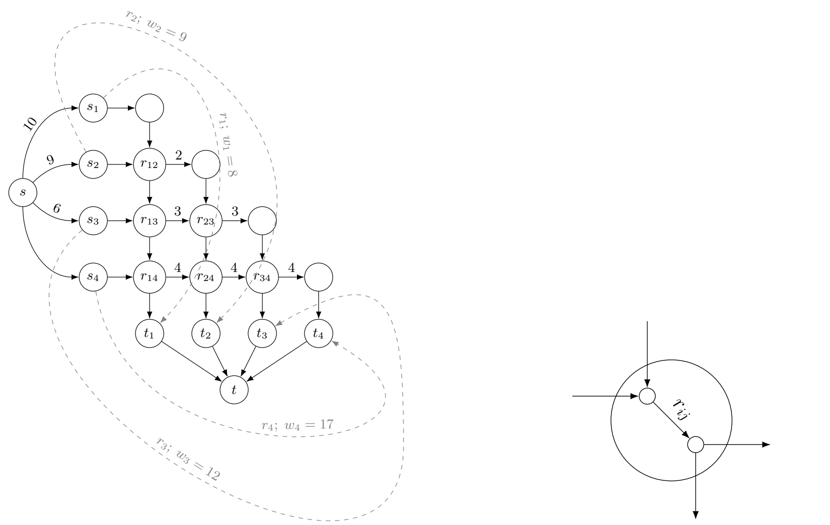
Extending the idea in Ackermann et al. [1], we reduce from the quadratic threshold game to DNC. To simplify our presentation, we assign positive integer weights to edges. Each weighted edge can be replaced by a chain of unit-length edges to obtain an unweighted graph.

Figure 1 illustrates the game with four players. We create $n(n+1)/2$ vertices arranged as a lower triangle. We use v_{ij} to denote the vertex at the i^{th} row (starting from top) and j^{th} column (starting from left) where $1 \le j \le i \le n$. The vertex v_{ij} is connected to $v_{i,j+1}$ with an edge of length i when $j < i$ and to $v_{i+1,j}$ with a unit-length edge when $i < n$. This design ensures that the shortest path from v_{i1} to v_{ni} is the right-down path. The resource r_{ij} is placed at the off-diagonal vertex v_{ij}, which can be implemented by splitting the vertex into two vertices connected by a unit-length edge with the delay function of r_{ij}. Note that this implies visiting a vertex v_{ij} incurs a distance of 1 where $i \neq j$. We then create vertices s_i and t_i for $1 \le i \le n$ with unit-length edges (s_i, v_{i1}) and (v_{ni}, t_i). We connect s_i to t_i with an edge of length w_i, which represents the resource r_i. Let b be the distance bound. We will determine the values of w_i and b later. The source s is connected to s_i with an edge of length $b - w_i - 1$. Vertices t_i are connected to the sink t via unit-length edges.

We define the following delay functions for edges associated with s or t:

$$d_{(s,\, s_i)}(x) = \mathbb{1}_{x \ge 2} \cdot (|\mathcal{N}| + 1)R \qquad d_{(t_i,\, t)}(x) = (|\mathcal{N}| - i)R$$
$$\text{where } R = \left(\sum_{r \in \mathcal{R}^{\text{in}} \cup \mathcal{R}^{\text{out}}} \max_{i \in \mathcal{N}} d_r(i) \right) + 1$$

We argue that player i chooses edges (s, s_i) and (t_i, t) in their best responses. Since R is greater than the maximum possible sum of delays of resources in the



(a) The graph structure

(b) Splitting the vertex containing resource r_{ij}

Fig. 1: The DNC instance corresponding to a four-player quadratic threshold game. The distance bound $b = 19$. Non-unit-length edges have labels to indicate their lengths. Dashed gray edges correspond to the S_i^{out} strategies.

threshold game, a player's best response must first optimize their choice of edges linked to s or t. If two players choose the edge (s, s_i), one of them can improve their latency by changing to an unoccupied edge $(s, s_{i'})$. Therefore, we can assume the i^{th} player chooses edge (s, s_i) WLOG. Player i can also decrease their latency by switching from (t_j, t) to (t_{j+1}, t) for any $j < i$ unless their strategy is limited by the distance bound when $j = i$.

Player i now has only two strategies from s_i to t_i due to the distance bound, corresponding to their strategies in the threshold game: (i) following the right–down path, namely $(s_i, v_{i1}, \cdots, v_{ii}, v_{i+1,i}, \cdots, v_{ni}, t_i)$, where they occupy resources corresponding to S_i^{in}; and (ii) using the edge (s_i, t_i), where they occupy the resource $S_i^{\text{out}} = \{r_i\}$. Clearly PNEs in this DNC correspond to PNEs in the original quadratic threshold game.

Now we determine the values of w_i and b. The shortest paths from s_i to t_i should be either the right-down path or the edge (s_i, t_i). This implies that $w_i = a_i + b_i + c_i$ where $a_i = i(i-1) + 1$ is the total length of horizontal edges, $b_i = n + 1 - i$ is the total length of vertical edges, and $c_i = n - 1$ is the total length of edges inside v_{ij} for resources r_{ij}. Hence $w_i = i(i-2) + 2n + 1$. The bound b should accommodate player n who has the longest path and is set as $b = w_n + 2 = n^2 + 3$.

Theorem 2. *Computing the best social welfare (i.e., minimal total delay) among PNEs of a DNC is* NP*-hard.*

Proof. We reduce from the strongly NP-complete 3-partition problem [10].

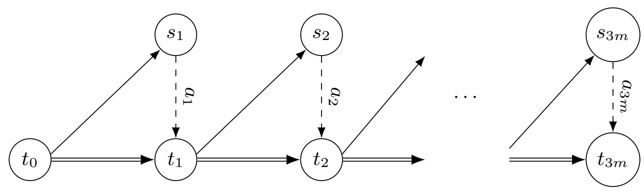


Fig. 2: Illustration of the DNC instance corresponding to a 3-partition problem. Double-line edges are slow edges, dashed edges are fast edges, and other edges have no delay. Non-unit-length edges have labels to indicate their lengths. Deciding whether the total delay can be bounded by $6m - 3$ is NP-complete.

In the *3-partition problem*, we are given a multiset of $3m$ positive integers $S = \{a_i \in \mathbb{Z}^+ \mid 1 \leq i \leq 3m\}$ and a number T such that $\sum a_i = mT$ and $T/4 < a_i < T/2$. The question Q_1 is: Can S be partitioned into m sets $S_1, \cdots, S_m$ such that $\sum_{a_i \in S_j} a_i = T$ for all $1 \leq j \leq m$? Note that due to the strong NP-completeness of 3-partition, we assume the numbers use unary encoding so that the DNC graph size is polynomial.

As in the proof of Theorem 1, we assign a weight $w_e \in \mathbb{Z}^+$ to each edge e. The DNC instance has two types of edges with non-zero delay: fast edge and slow edge, with delay functions $d_{\text{fast}}(x) = \mathbb{1}_{x \geq 1} + 2\mathbb{1}_{x \geq 2}$ and $d_{\text{slow}}(x) = 2$.

As illustrated in Fig. 2, for each integer a_i, we create a pair of vertices (s_i, t_i) connected by a fast edge with $w_{(s_i, t_i)} = a_i$. We create a new vertex t_0 as the source while using t_{3m} as the sink. For $0 \leq i < 3m$, we connect t_i to t_{i+1} by a unit-length slow edge and t_i to s_{i+1} by a unit-length edge without delay. There are m players who can choose paths with length bounded by $b = T + 3m$.

We ask the question Q_2: Is there a PNE in the above game where the total delay is no more than $m(6m-3)$? Each player prefers an unoccupied fast edge to a slow edge but also prefers a slow edge to an occupied fast edge due to the above delay functions. Since $T/4 < a_i < T/2$, the best response of a player contains either 2 or 3 fast edges, contributing $6m-2$ or $6m-3$ to the total delay in either case. Best social welfare of $m(6m-3)$ is only achieved when every player chooses 3 fast edges, which also means that their choices together constitute a partition of the integer set S in Q_1. Therefore, Q_2 and Q_1 have the same answer.

Remark The optimal global welfare of any "centralized" solution (where players cooperate to minimize total delay instead of selfishly minimizing their own delay) achieves $m(6m-3)$ if and only if the original 3-partition problem has a solution. Hence we also have the following theorem:

Theorem 3. *Computing the optimal global welfare of pure strategies in DNC is* NP*-hard.*

Theorem 4. *Computing the worst social welfare (i.e., maximal total delay) among PNEs of a DNC is* NP*-hard.*

Proof. We build on the proof of Theorem 2. We create a new vertex s as the source and connect s to t_0 and s_i where $1 \leq i \leq 3m$:

$$\begin{aligned} w_{(s,\,t_0)} &= 1 & d_{(s,\,t_0)}(x) &= \mathbb{1}_{x \geq m+1} \cdot R \\ w_{(s,\,s_i)} &= T + i - a_i + 1 & d_{(s,\,s_i)}(x) &= \mathbb{1}_{x \geq 2} \cdot R \end{aligned} \qquad \text{where R = 9m + 2}$$

The delay functions on fast and slow edges are changed to $d_{\text{fast}}(x) = 2\mathbb{1}_{x\geq 2} + 2\mathbb{1}_{x\geq 3}$ and $d_{\text{slow}}(x) = 3$.

There are $4m$ players in this game with a distance bound $b = T + 3m + 1$. Since R is greater than the delay on any path from s_i or t_0 to the sink, we can assume WLOG that player i choose $(s,\, s_i)$ where $1 \leq i \leq 3m$, and players $3m+1, \cdots, 4m$ all choose $(s,\, t_0)$. The first $3m$ players generate a total delay of $D_0 = d_{\text{slow}} \cdot 3m(3m-1)/2 = 9m(3m-1)/2$ where player i occupies one fast edge and $3m-i$ slow edges. Each of the last m players occupies 2 or 3 fast edges in their best response. Occupying one fast edge incurs 4 total delay because one of the first $3m$ players also uses that edge. Therefore, the each of last m players contributes $9m+2$ or $9m+3$ to the total delay. We ask the question Q_3: Is there a PNE where the total delay is at least $D_0 + m(9m+3)$? From our analysis, we can see that Q_3 and Q_1 have the same answer.

2.1 Distance-bounded network congestion game with default action

As we have discussed, we formulate capability restriction as limiting the size of the programs accessible to a player. In this section, we propose a variant of DNC where we define a DSL to compactly represent the strategies. We will also show that the size of a program equals the length of the path generated by the program, which can be much smaller than the number of edges in the path. The new game, called *distance-bounded network congestion game with default action (DncDa)*, requires that each vertex except the source or sink has exactly one outgoing zero-length edge as its default action. All other edges have unit length. A strategy in this game can be compactly described by the actions taken at *divergent points* where a unit-length edge is followed.

Definition 2. *An instance of DncDa is a tuple* $G = (\mathcal{V},\, \mathcal{E},\, \mathcal{N},\, s,\, t,\, b,\, (d_e)_{e\in\mathcal{E}},\, (w_e)_{e\in\mathcal{E}})$ *where:*

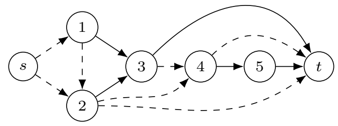
- $w_e \in \{0,\, 1\}$ *is the length of edge* e.
- *All other symbols have the same meaning as in Definition 1.*

Moreover, we require the following properties:

- *A default action, denoted as* $\mathrm{DA}(\cdot)$*, can be defined for every non-source, non-sink vertex* $v \in \mathcal{V}/\{s,\, t\}$ *such that:*

$$\Big(v,\, \mathrm{DA}(v)\Big) \in \mathcal{E}, \quad w_{(v,\,\mathrm{DA}(v))} = 0,$$
$$\forall u \in \mathcal{V}/\{\mathrm{DA}(v)\} :\ (v,\, u) \in \mathcal{E} \implies w_{(v,\,u)} = 1$$

- *Edges from the source have unit length:* $\forall v \in \mathcal{V} :\ (s,\, v) \in \mathcal{E} \implies w_{(s,\,v)} = 1$



(a) Example graph structure. Solid arrows are default edges and dashed arrows are unit-length edges.

```
if (u == s) { return 2; } else {
    if (u == 3) { return 4; } else {
        return DA(u);
    }
}
```

(b) The shortest program to represent the strategy (s, 2, 3, 4, 5, t).

Fig. 3: An example of the DncDa game and a program to represent a strategy.

– *The subgraph of zero-length edges is acyclic. Equivalently, starting from any non-source vertex, one can follow the default actions to reach the sink.*

The strategy space of a single player contains all $s-t$ simple paths whose length does not exceed b:

$$\mathcal{L}_b \overset{def}{=} \left\{ (p_0,\, \ldots,\, p_k) \;\middle|\; \begin{array}{l} p_0 = s,\, p_k = t,\, (p_i,\, p_{i+1}) \in \mathcal{E}, \sum_{i=0}^{k-1} w_{(p_i,\, p_{i+1})} \leq b, \\ p_i \neq p_j \textit{ for } i \neq j \end{array} \right\}$$

Note that the strategy spaces are strictly monotonically increasing up to the longest simple $s-t$ path. This is because for any path p whose length is $b \geq 2$, we can remove the last non-zero edge on p and follow the default actions to arrive at t, which gives a new path with length $b-1$. Formally, we have:

Property 1. Let $\bar{b}$ be the length of the longest simple $s-t$ path in a DncDa instance. For $1 \leq b < \bar{b}$, $\mathcal{L}_b \subsetneq \mathcal{L}_{b+1}$

We define a Domain Specific Language (DSL) with the following context-free grammar [12] to describe the strategy of a player:

$$\begin{array}{rcl} \texttt{Program} & \rightarrow & \textbf{return}\ \texttt{DA}(u); \\ & & \mid\ \textbf{if}\ (u\ \texttt{==}\ \texttt{V})\ \{\textbf{return}\ \texttt{V};\}\ \textbf{else}\ \{\texttt{Program}\} \\ \texttt{V} & \rightarrow & v \in \mathcal{V} \end{array}$$

A program p in this DSL defines a computable function $f_p : \mathcal{V} \mapsto \mathcal{V}$ with semantics similar to the C language where the input vertex is stored in the variable u, as illustrated in Fig. 3. The strategy corresponding to the program p is a path $(c_0,\, \ldots,\, c_k)$ from s to t where:

$$c_0 = s \qquad c_{i+1} = f_p(c_i) \text{ for } i \geq 0 \text{ and } c_i \neq t \qquad k = i \text{ if } c_i = t$$

We define the capability of a player as the maximum size of programs that they can use. The size of a program is the depth of its parse tree. Due to the properties of DncDa, the shortest program that encodes a path from s to t specifies the edge chosen at all divergent points in this path. The size of this program equals the length of the path. Hence the distance bound in the game configuration specifies the capability of each player in the game. To study the game outcome under different player capability constraints, we study DncDa instances with different values of b.

We state hardness results for DncDa. Their proofs are similar to those for DNC except that we need to redesign the edge weights to conform to the requirements on default action. The proof details are given in Appendix A.

Theorem 5. *DNCDA is* PLS*-complete.*

Theorem 6. *Computing the best social welfare (i.e., minimal total delay) among PNEs of a DNCDA is* NP*-hard.*

Theorem 7. *Computing the worst social welfare (i.e., maximal total delay) among PNEs of a DNCDA is* NP*-hard.*

Theorem 8. *Computing the optimal global welfare of pure strategies in DNCDA is* NP*-hard.*

3 Impact of player capability on social welfare in DNCDA

We first introduce four capability preference properties for general games. Given a game with a finite hierarchy of player capabilities, we use $\mathcal{L}_b$ to denote the strategy space when player capability is bounded by b. Assuming the maximal capability is $\bar{b}$ (which is the longest $s-t$ simple path in DNCDA), we have $\mathcal{L}_b \subsetneq \mathcal{L}_{b+1}$ for $1 \le b < \bar{b}$ (see Property 1). We use $\text{Equil}(b) \subseteq \mathcal{L}_b^n$ to denote the set of all PNEs at the capability level b. We define $W_b^+ \overset{\text{def}}{=} \max_{\boldsymbol{s}\in\text{Equil}(b)} W(\boldsymbol{s})$ to be the best social welfare at equilibrium and $W_b^- \overset{\text{def}}{=} \min_{\boldsymbol{s}\in\text{Equil}(b)} W(\boldsymbol{s})$ the worst social welfare.

Definition 3. *A game is capability-positive if social welfare at equilibrium cannot decrease as players become more capable, i.e.,* $\forall 1 \le b < \bar{b}$, $W_b^+ \le W_{b+1}^-$.

Definition 4. *A game is max-capability-preferred if the worst social welfare at equilibrium under maximal player capability is at least as good as any social welfare at equilibrium under lower player capability, i.e.,* $\forall 1 \le b < \bar{b}$, $W_b^+ \le W_{\bar{b}}^-$.

Note that max-capability-preferred is a weaker condition than capability-positive. We then define analogous properties for games where less capable players lead to better outcomes:

Definition 5. *A game is capability-negative if social welfare at equilibrium cannot increase as players become more capable, i.e.,* $\forall 1 \le b < \bar{b}$, $W_{b+1}^+ \le W_b^-$.

Definition 6. *A game is min-capability-preferred if the worst social welfare at equilibrium under minimal player capability is at least as good as any social welfare at equilibrium under higher player capability, i.e.,* $\forall b \ge 2$, $W_b^+ \le W_1^-$.

Our goal is to identify games that guarantee these properties. Since solving equilibria for general DNCDA is computationally hard (Section 2.1), we focus on a restricted version of DNCDA where all edges share the same delay function; formally, we consider the case $\forall e \in \mathcal{E} : d_e(\cdot) = d(\cdot)$ where $d(\cdot)$ is non-negative and non-decreasing. We call this game *distance-bounded network congestion game with default action and shared delay (DNCDAS)*. We aim to find conditions on $d(\cdot)$ under which the properties hold *universally* (i.e., for all network configurations of DNCDAS). Table 1 summarizes the results.

	DncDaS (Section 3)	GMG (Sections 4 and 5)
Resource layout	On a directed graph	On parallel horizontal lines
Strategy space	Paths from s to t	Piecewise-constant functions
Delay (payoff)	Non-negative non-decreasing	$r_g(\cdot)$ positive, $r_m(\cdot)$ negative
capability-positive	$d(\cdot)$ is a constant function	$r_g(\cdot), r_m(\cdot)$ are constant functions
max-capability-preferred	$d(\cdot)$ is a constant function	$w(x) = xr_g(x)$ attains maximum at $x = n$
capability-negative	$d(\cdot)$ is the zero function	Never
min-capability-preferred	$d(\cdot)$ is the zero function	Never

Table 1: Necessary and sufficient conditions on the delay or payoff functions such that the capability preference properties hold universally.

Theorem 9. *DncDaS is universally capability-positive if and only if $d(\cdot)$ is a constant function.*

Proof. If $d(\cdot)$ is a constant function, the total delay achieved by a strategy is not affected by the load condition of each edge (thus not affected by other players' strategies). So each player's strategy in any PNE is the one in $\mathcal{L}_b$ that minimizes the total delay under the game layout. Denote this minimum delay as $\delta(b)$. For any $b \geq 1$, we have $\mathcal{L}_b \subseteq \mathcal{L}_{b+1}$, so $\delta(b) \geq \delta(b+1)$. And for any $\boldsymbol{s} \in \text{Equil}(b)$, $W(\boldsymbol{s}) = -n\delta(b)$. Hence $W_b^+ \leq W_{b+1}^-$.

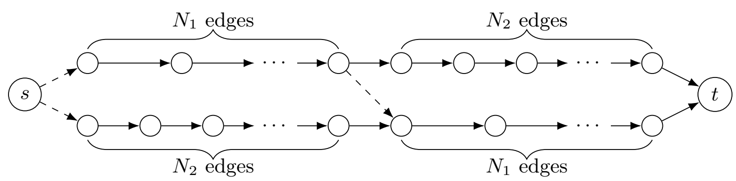
If $d(\cdot)$ is not a constant function, we show that there exists an instance of DncDaS with delay function $d(\cdot)$ that is not capability-positive. Define $v = \min\{x \mid d(x) \neq d(x+1)\}$. It follows that $d(v') = d(v)$ for all $v' \leq v$. We consider the cases $d(v) = 0$ and $d(v) > 0$ separately.

Case 1: $d(v) > 0$ Denote $\rho = \frac{d(v+1)}{d(v)}$. Since $d(\cdot)$ is non-decreasing, $\rho > 1$. We construct a game with the network layout in Fig. 4a with $n = v+1$ players. We will show that $W_1^+ > W_2^-$.

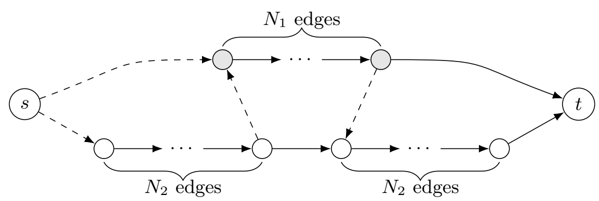
First, it is easy to see that the PNEs when $b = 1$ are a players take the upper path and $v + 1 - a$ players take the lower path, where $1 \leq a \leq v$. All PNEs achieves a social welfare of $W_1 = -(v+1)(N_1 + N_2 + 3)d(v)$.

We set $N_1 = \lfloor \frac{1}{\rho - 1} \rfloor$ and $N_2 = \lfloor (N_1 + 2)\rho \rfloor - 1$. When $b = 2$, one PNE is that all players choose the path from upper left to lower right using the crossing edge in the middle due to our choice of N_1 and N_2. Its social welfare $W_2 = -(v+1)(2N_1+3)d(v+1)$. One can check that $W_1 > W_2$, hence $W_1^+ > W_2^-$. Appendix B presents more details.

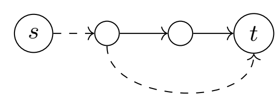
Case 2: $d(v) = 0$ We construct a game with the network layout in Fig. 4b where there are $2v$ players. With $b = 1$, half of the players choose the upper path and the others choose the lower path, which has a social welfare $W_1 = 0$. With $b = 2$, a PNE is: (i) v players take the path (s, N_1 edges, lower right N_2 edges, t);



(a) Counterexample for capability-positive in the case $d(v) > 0$.



(b) Counterexample for capability-positive in the case $d(v) = 0$. Edges between filled nodes have non-zero delay in an equilibrium when $b = 2$.



(c) Counterexample for capability-negative.

Fig. 4: Counterexamples when $d(\cdot)$ does not meet the conditions. Dashed arrows denote unit-length edges and solid arrows denote zero-length edges (default action). Every edge shares the same delay function $d(\cdot)$.

and (ii) the other v players take the path (s, lower left N_2 edges, N_1 edges, t). We choose N_1 and N_2 to be positive integers that satisfy $\frac{N_2+1}{N_1} > \frac{d(2v)}{d(v+1)}$. The social welfare $W_2 = -2vN_1 d(2v) < W_1$. Hence $W_1^+ > W_2^-$.

Remark The proof for the sufficient condition also holds when different edges have different delay functions. So the following statement is also true: DncDa is universally capability-positive if all edges have constant delay functions.

Theorem 10. *DncDaS is universally max-capability-preferred if and only if $d(\cdot)$ is a constant function.*

Proof. The "if" part follows from Theorem 9 since a capability-positive game is also max-capability-preferred. The constructed games in the proof of Theorem 9 also serve as the counterexamples to prove the "only if" part.

Theorem 11. *DncDaS is universally capability-negative if and only if $d(\cdot)$ is the zero function.*

Proof. If $d(\cdot) = 0$, then all PNEs have welfare 0, which implies capability-negative. If $d(\cdot)$ is not the zero function, denote $v = \min\{x \mid d(x) \neq 0\}$. We construct a game with the network layout shown in Fig. 4c with $n = v$ players. When $b = 1$, all players use the only strategy with a social welfare $W_1 = -3vd(v)$. When $b = 2$: if $v = 1$, the player will choose both dashed paths and achieves $W_2 = -2d(1)$; if $v \geq 2$, the players will only experience delay on the first edge by splitting between the default path and the shortcut dashed path, which achieves a welfare $W_2 = -vd(v)$. In both cases, the game is not capability-negative since $W_1^- \le W_1 < W_2 \le W_2^+$.

The same argument can also be used to prove the following result:

Theorem 12. *DncDaS is universally min-capability-preferred if and only if $d(\cdot)$ is the zero function.*

4 Gold and Mines Game

In this section, we introduce a particular form of DncDa called Gold and Mines Game (GMG). It provides a new perspective on how to define the strategy space hierarchy in congestion games. It also enables us to obtain additional characterizations of how social welfare at equilibrium varies with player capability. Intuitively, as shown in Fig. 5, a GMG instance consists of a few parallel horizontal lines and two types of resources: gold and mine. Resources are placed at distinct horizontal locations on the lines. A player's strategy is a piecewise-constant function to cover a subset of resources. The function is specified by a program using if-statements.

Definition 7. *An instance of GMG is a tuple $G = (\mathcal{E}, K, \mathcal{N}, r_g, r_m, b)$ where:*

- *$\mathcal{E}$ is the set of resources. Each resource $e \in \mathcal{E}$ is described by a tuple (x_e, y_e, α_e), where (x_e, y_e) denotes the position of the resource in the x-y plane, and $\alpha_e \in \{gold, mine\}$ denotes the type of the resource. Each resource has a distinct value of x, i.e. $x_e \neq x_{e'}$ for all $e \neq e'$.*
- *$K \in \mathbb{N}$ is the number of lines the resources can reside on. All resources are located on lines $y = 0$, $y = 1$, ..., $y = K-1$, i.e. $\forall e, y_e \in \{0, 1, \ldots, K-1\}$.*
- *$\mathcal{N} = \{1, \ldots, n\}$ denotes the set of players.*
- *$r_g : \mathbb{N} \mapsto \mathbb{R}^+$ is the payoff function for gold. r_g is a positive function.*
- *$r_m : \mathbb{N} \mapsto \mathbb{R}^-$ is the payoff function for mine. r_m is a negative function.*
- *$b \in \mathbb{N}$ is the level in the strategy space hierarchy defined by the domain-specific language $\mathcal{L}$ (defined below). The strategy space is then $\mathcal{L}_b$.*

The strategy s_i of player i is represented by a function $f_i(\cdot)$ that conforms to a domain-specific language $\mathcal{L}$ with the following grammar:

$$\begin{array}{lcl} \text{Program} & \rightarrow & \textbf{return}\ \text{C};\ \mid\ \textbf{if}\ (x < t)\ \{\textbf{return}\ \text{C};\}\ \textbf{else}\ \{\text{Program}\} \\ \text{C} & \rightarrow & 0 \mid 1 \mid \ldots \mid K-1 \\ \text{t} & \in & \mathbb{R} \end{array}$$

This DSL defines a natural strategy space hierarchy by restricting the number of if-statements in the program. A program with $b-1$ if-statements represents a piecewise-constant function with at most b segments. We denote $\mathcal{L}_b$ as the level b strategy space which includes functions with at most $b-1$ if-statements.

Player i *covers* the resources that their function f_i passes: $E_i = \{e \mid f_i(x_e) = y_e\}$. The load on each resource is the number of players that covers it: $x_e = |\{i \mid e \in E_i\}|$. Each player's payoff is $u_i = \sum_{e \in E_i} r_e(x_e)$, where r_e is either r_g or r_m depending on the resource type. The social welfare is $W(\boldsymbol{s}) = \sum_{i \in \mathcal{N}} u_i$.

Note that GMG can be represented as a DncDa (Appendix C). As a result, the problem of computing a PNE in a GMG belongs to PLS.

5 Impact of player capability on social welfare in GMG

We revisit the question of how player capability affects the social welfare at equilibrium in the context of GMG. We consider the same capability preference properties as in Section 3. The results are summarized in Table 1.

Theorem 13. *GMG is universally capability-positive if and only if both* $r_g(\cdot)$ *and* $r_m(\cdot)$ *are constant functions.*

The idea of the proof is similar to that of Theorem 9. The construction of the counterexamples for the "only if" part requires some careful design on the resource layout. Appendix D presents the proof.

Theorem 14. *Define welfare function for gold as* $w_g(x) \stackrel{def}{=} x \cdot r_g(x)$*. GMG is universally max-capability-preferred if and only if* w_g *attains its maximum at* $x = n$ *(i.e.* $\max_{x \le n} w_g(x) = w_g(n)$*), where* n *is the number of players.*

Proof. We first notice that there is only one PNE when $b = \bar{b}$, which is all players cover all gold and no mines. This is because r_g is a positive function and r_m is a negative function, and since all x_e's are distinct, each player can cover an arbitrary subset of the resources when $b = \bar{b}$. So $W_{\bar{b}} = nM_g r_g(n)$ where M_g is the number of gold in the game.

If $\max_{x \le n} w_g(x) = w_g(n)$, we show that $W_{\bar{b}}$ is actually the maximum social welfare over all possible strategy profiles of the game. For any strategy profile $\boldsymbol{s}$ of the game, the social welfare

$$W(\boldsymbol{s}) = \sum_{i \in \mathcal{N}} \sum_{e \in E_i} r_e(x_e) = \sum_{e \in \mathcal{E}} x_e \cdot r_e(x_e) \le \sum_{e \in \mathcal{E}_g} n \cdot r_g(n) = nM_g r_g(n) = W_{\bar{b}}.$$

Therefore, the game is max-capability-preferred.

If $\max_{x \le n} w_g(x) > w_g(n)$, denote $n' = \arg\max_{x \le n} w_g(x)$, $n' < n$. We construct a game with the corresponding $r_g(\cdot)$ that is not max-capability-preferred. The game has n players, $K = n$ lines, and each line has one gold. The only PNE in $\mathrm{Equil}(\bar{b})$ is all players cover all gold, which achieves a social welfare of $W_{\bar{b}} = n \cdot w_g(n)$. When $b = n'$, one PNE is each player covers n' gold, with each gold covered by exactly n' players. This can be achieved by letting player i cover the gold on lines $\{y = (j \mod n)\}_{j=i}^{i+n'-1}$. To see why this is a PNE, notice that any player's alternative strategy only allows them to switch to gold with load larger than n'. For all $x > n'$, since $w_g(n') \ge w_g(x)$, $r_g(n') > r_g(x)$. So such change of strategy can only decrease the payoff of the player. The above PNE achieves a social welfare $W_{n'} = n \cdot w_g(n') > W_{\bar{b}}$, so the game is not max-capability-preferred.

Theorem 15. *For any payoff functions* $r_g(\cdot)$ *and* $r_m(\cdot)$*, there exists an instance of GMG where min-capability-preferred does not hold (therefore capability-negative does not hold either).*

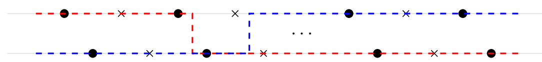


Fig. 5: Resource layout for the alternating ordering game. Each dot (resp. cross) is a gold (resp. mine). The dashed lines represent a PNE when $b = 2$ (with $-2 + \rho < \mu < -\rho$).

Proof. It is trivial to construct such a game with mines. Here we show that for arbitrary $r_g(\cdot)$, we can actually construct a game with only gold that is not min-capability-preferred.

Let $r_{\min} = \min_{x \leq n} r_g(x)$ and $r_{\max} = \max_{x \leq n} r_g(x)$. We construct a game with $K = 2$ lines and $N + 1$ gold where $N > \frac{r_{\max}}{r_{\min}}$. In the order of increasing x, the first N gold is on $y = 0$ and the final gold is on $y = 1$. When $b = 1$, for an arbitrary player, denoting the payoff of choosing $y = 0$ (resp. $y = 1$) as r_0 (resp. r_1). Then $r_0 = \sum_{e \in \mathcal{E}_0} r_g(x_e) \geq \sum_{e \in \mathcal{E}_0} r_{\min} = N r_{\min} > r_{\max} \geq r_1$, where $\mathcal{E}_0$ is the set of resources on $y = 0$. So all the players will choose $y = 0$ in the PNE. The social welfare is $W_1 = nNr_g(n)$. When $b = 2$, all the players will choose to cover all the gold in the PNE. So the social welfare is $W_2 = n(N+1)r_g(n) > W_1$. Therefore, the game is not min-capability-preferred.

6 Case study: alternating ordering game

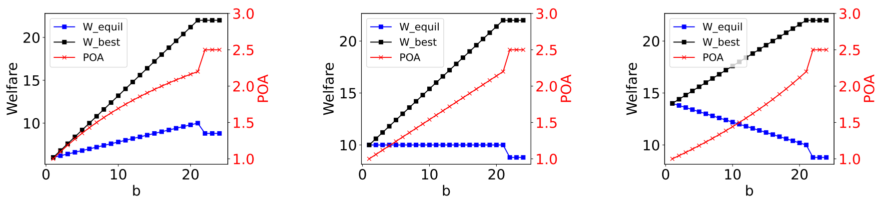
In this section, we present a special form of GMG called the alternating ordering game. We derive exact expressions of the social welfare at equilibrium with respect to the capability bound. The analysis provides insights on the factors that affect the trend of social welfare over player capability.

Definition 8. *The alternating ordering game is a special form of the GMG, with $n = 2$ players and $K = 2$ lines. The layout of the resources follows an alternating ordering of gold and mines as shown in Fig. 5. Each line has M mines and $M+1$ gold. The payoff functions satisfy $0 < r_g(2) < \frac{r_g(1)}{2}$ (reflecting competition when both players occupy the same gold) and $r_m(1) = r_m(2) < 0$. WLOG, we consider normalized payoff where $r_g(1) = 1, r_g(2) = \rho, 0 < \rho < \frac{1}{2}, r_m(1) = r_m(2) = \mu < 0$.*

Let's consider the cases $b = 1$ and $b = 2$ to build some intuitive understanding. When $b = 1$, the PNE is that each player covers one line, which has social welfare $W_1 = 2M + 2M\mu + 2$. When $b = 2$ (and $-2 + \rho < \mu < -\rho$), one PNE is shown in Fig. 5, where the players avoid one mine but cover one gold together, which has social welfare $W_2 = W_1 - 1 - \mu + 2\rho$. Whether the social welfare at $b = 2$ is better depends on the sign of $2\rho - \mu - 1$. In fact, we have the following general result:

Theorem 16. *If $-2 + \rho < \mu < -\rho$, then for any level b strategy space $\mathcal{L}_b$, all PNEs have the same social welfare*

$$W_{\text{Equil}}(b) = \begin{cases} (2M+1)(1+\mu) + 2(1-\rho) + (2\rho - \mu - 1)b & \text{if } b \leq 2M+1 \\ (4M+4)\rho & \text{if } b \geq 2M+2 \end{cases}.$$



(a) $\mu = -0.8$ ($\mu < 2\rho - 1$) (b) $\mu = -0.6$ ($\mu = 2\rho - 1$) (c) $\mu = -0.4$ ($\mu > 2\rho - 1$)

Fig. 6: W_{Equil}, W_{best}, POA varying with b. $M = 10, \rho = 0.2$.

The full proof is lengthy and involves analyses of many different cases (Appendix E). We present the main idea here.

Proof idea. We make three arguments for this proof: (i) Any function in a PNE must satisfy some specific form indicating where it can switch lines; (ii) Any PNE under $\mathcal{L}_b$ must consist of only functions that use exactly b segments; and (iii) For any function with b segments that satisfies the specific form, the optimal strategy for the other player always achieves the same payoff.

Remark $-2 + \rho < \mu < -\rho$ is in fact a necessary and sufficient condition for all PNEs having the same social welfare for any b and M (see Appendix E.4).

Depending on the sign of $2\rho - \mu - 1$, $W_{\text{Equil}}(b)$ can increase, stay the same, or decrease as b increases until $b = 2M + 1$. $W_{\text{Equil}}(b)$ always decreases at $b = 2M + 2$ and stays the same afterwards. Figure 6 visualizes this trend. Figure 7 summarizes how the characteristics of the PNEs varies in the ρ-μ space.

Price of Anarchy The price of anarchy (POA) [13] is the ratio between the best social welfare achieved by any centralized solution and the worst welfare at equilibria: $\text{POA}(b) = \frac{W_{\text{best}}(b)}{W_{\text{Equil}}(b)}$. We can show that the best centralized social welfare is $W_{\text{best}}(b) = 2M + 2 + \mu \cdot \max(2M + 1 - b, 0)$ (Appendix E.3). Hence

$$\text{POA}(b) = \begin{cases} 1 + \frac{(1-2\rho)(b-1)}{2M+2+2M\mu+(2\rho-\mu-1)(b-1)} & \text{if } b \leq 2M + 1 \\ \frac{1}{2\rho} & \text{if } b \geq 2M + 2 \end{cases},$$

$\text{POA}(b)$ increases with b up to $b = 2M + 2$, then stays the same (Fig. 6).

Interpretation There are two opposing factors that affect whether increased capability is beneficial for social welfare or not. With increased capability, players can improve their payoff in a non-competitive way (e.g. avoiding mines), which is always beneficial for social welfare; they can also improve payoff in a competitive way (e.g. occupying gold together), which may reduce social welfare. The joint effect of the two factors determines the effect of increasing capability.

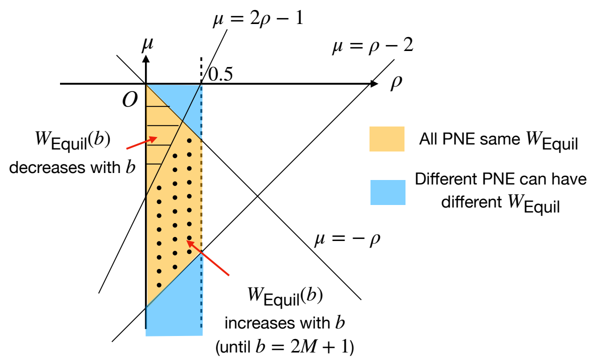


Fig. 7: Characteristics of PNEs over the ρ-μ landscape.

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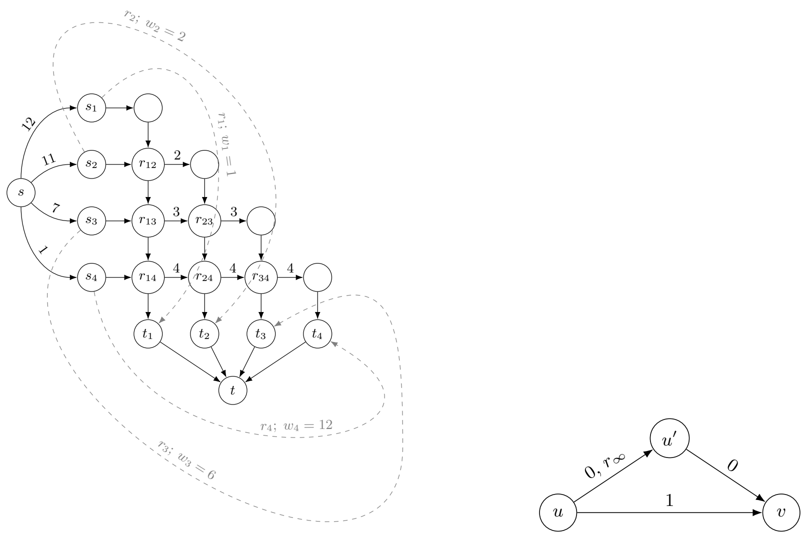
A Proofs of the DNCDA hardness results

A.1 Proof of Theorem 5

Theorem 5. *DNCDA is* PLS*-complete.*



(a) The graph structure

(b) A gadget to add a default action for u

Fig. 8: Illustration of the DNCDA instance corresponding to a four-player quadratic threshold game. The distance bound is $b = 13$. Non-zero-length edges have labels to indicate their lengths.

Proof. The best response of a player in DNCDA can be computed in polynomial time similarly to DNC. Now we prove its PLS-completeness by presenting a reduction from the quadratic threshold game. We modify the network layout in the proof of Theorem 1 as follows. We assign zero length to the vertical edges $(v_{ij},\, v_{i+1,j})$, edges for r_{ij}, and edges in the set $\cup_{1\le i\le n}\{(s_i,\, v_{i1}),\, (v_{ni},\, t_i),\, (t_i,\, t)\}$. We also redefine the lengths of some other edges: $w_{(s_i,\, t_i)} = i(i-1) + \mathbb{1}_{i=1}$ and $w_{(s,\, s_i)} = b - w_{(s_i,\, t_i)}$. The distance bound is $b = n(n-1)+1$. Note that after replacing weighted edges with a chain of unit-length edges to build the DNCDA network, some vertices will only have one unit-length outgoing edge, which violates the requirements of default action. In this case, we add auxiliary vertices with zero-length edges and a resource r_∞ that has sufficiently large delay to disincentivize any player from taking the zero-length auxiliary edges. Figure 8 illustrates this construction. In this DNCDA instance, player i will choose edges $(s,\, s_i)$ and $(t_i,\, t)$. They then choose between the right-down path from s_i to t_i or the edge $(s_i,\, t_i)$ which correspond to the two strategies in the quadratic threshold game respectively.

A.2 Proof of Theorem 6 and Theorem 7

Theorem 6. *Computing the best social welfare (i.e., minimal total delay) among PNEs of a* DncDa *is* NP*-hard.*

Proof. We adopt the reduction given in the proof of Theorem 2. We modify the edge weights as $w_{(t_i,\, s_{i+1})} = 1$, $w_{(s_i,\, t_i)} = a_i - 1$, and $w_{(t_i,\, t_{i+1})} = 0$, as illustrated in Fig. 9. The distance bound is $b = T$. All other argument in the original proof follows.

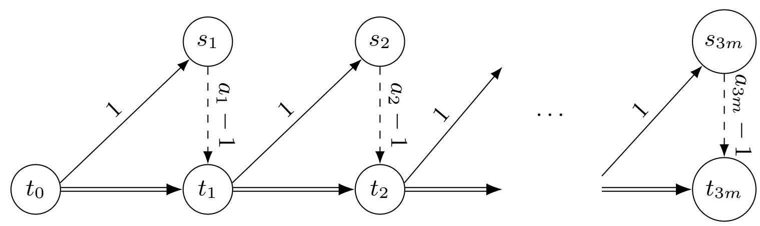


Fig. 9: Illustration of the DncDa instance corresponding to a 3-partition problem. Double-line edges are slow edges, dashed edges are fast edges, and other edges have no delay. Non-unit-length edges have labels to indicate their lengths. Deciding whether the total delay can be bounded by $6m - 3$ is NP-complete.

With similar adoption of the proof of Theorem 4, we can also prove Theorem 7, which claims the NP-hardness of computing the worst social welfare at equilibrium.

B Detailed proof of Theorem 9

Theorem 9. *DncDaS is universally capability-positive if and only if $d(\cdot)$ is a constant function.*

Proof. If $d(\cdot)$ is a constant function, the total delay achieved by a strategy is not affected by the load condition of each edge (thus not affected by other players' strategies). So each player's strategy in any PNE is the one in $\mathcal{L}_b$ that minimizes the total delay under the game layout. Denote this minimum delay as $\delta(b)$. For any $b \geq 1$, we have $\mathcal{L}_b \subseteq \mathcal{L}_{b+1}$, so $\delta(b) \geq \delta(b+1)$. And for any $\boldsymbol{s} \in \mathrm{Equil}(b)$, $W(\boldsymbol{s}) = -n\delta(b)$. Hence $W_b^+ \leq W_{b+1}^-$.

If $d(\cdot)$ is not a constant function, we show that there exists an instance of DncDaS with delay function $d(\cdot)$ that is not capability-positive. Define $v = \min\{x \mid d(x) \neq d(x+1)\}$. It follows that $d(v') = d(v)$ for all $v' \leq v$. We consider the cases $d(v) = 0$ and $d(v) > 0$ separately.

Case 1: $d(v) > 0$ Denote $\rho = \frac{d(v+1)}{d(v)}$. Since $d(\cdot)$ is non-decreasing, $\rho > 1$.

We construct a game with the network layout in Fig. 4a with $n = v + 1$ players. This game is a counterexample for the capability-positive property with $b = 1$ (i.e., $W_1^+ > W_2^-$).

First, it is easy to see that the PNEs when $b = 1$ are a players take the upper path and $v + 1 - a$ players take the lower path, where $1 \leq a \leq v$. All PNEs achieves a social welfare of $W_1 = -(v + 1)(N_1 + N_2 + 3)d(v)$.

We set the constants $N_1 = \lfloor \frac{1}{\rho - 1} \rfloor$ and $N_2 = \lfloor (N_1 + 2)\rho \rfloor - 1$, which ensures $N_1 > \frac{1}{\rho-1} - 1$ and $(N_1 + 2)\rho - 2 < N_2 \leq (N_1 + 2)\rho - 1$. We claim that one PNE when $b = 2$ is that all players choose the path from upper left to lower right using the switching edge in the middle. Under this strategy profile, each player has a total delay of $\delta = (2N_1 + 3)d(v + 1)$. This is an equilibrium because if any player changes their strategy to the upper or the lower horizontal path (the only two alternative strategies) the new total delay is $\delta' = (N_1 + 1)d(v + 1) + (N_2 + 2)d(v) > \delta$ because $\frac{\delta' - \delta}{d(v)} = N_2 - \big((N_1 + 2)\rho - 2\big) > 0$. The social welfare is $W_2 = -(v + 1)(2N_1 + 3)d(v + 1)$. Note that

$$\begin{aligned}\frac{W_1 - W_2}{(v + 1)d(v)} &= (N_1 + 1)(\rho - 1) - \big(N_2 + 2 - (N_1 + 2)\rho\big) \\ &> (\frac{1}{\rho - 1} - 1 + 1)(\rho - 1) - \big((N_1 + 2)\rho - 1 + 2 - (N_1 + 2)\rho\big) = 0.\end{aligned}$$

Hence $W_1^+ \geq W_1 > W_2 \geq W_2^-$.

Case 2: $d(v) = 0$ We construct a game with the network layout in Fig. 4b where there are $2v$ players. With $b = 1$, half of the players choose the upper path and the others choose the lower path, which has a social welfare $W_1 = 0$.

With the bound $b = 2$, we consider this strategy: (i) v players take the path (s, N_1 edges, lower right N_2 edges, t); and (ii) the other v players take the path (s, lower left N_2 edges, N_1 edges, t). We choose N_1 and N_2 to be positive integers that satisfy $\frac{N_2+1}{N_1} > \frac{d(2v)}{d(v+1)}$. There are $2v$ players occupying the N_1 edges that incur a delay of $N_1 d(2v)$ on each player. The social welfare $W_2 = -2vN_1 d(2v) < W_1$. The above strategy is an equilibrium because for each player the alternative strategy to avoid the N_1 congestion edges is to take the lower path (s, N_2 edges, N_2 edges, t) which has a delay of $(N_2 + 1)d(v + 1) > N_1 d(2v)$. Hence $W_1^+ \geq W_1 > W_2 \geq W_2^-$.

C Representing GMG as DncDa

To represent a GMG as a DncDa, we first order all resources by increasing order in x_e, such that $x_1 < x_2 < \cdots < x_{|\mathcal{E}|}$. Then the network in the corresponding DncDa has vertices:

- $v_{i,j}$ at $(\frac{x_i + x_{i+1}}{2}, j)$ for all $i = 1, \ldots, |\mathcal{E}| - 1$ and $j = 0, \ldots, K - 1$.
- $v_{0,j}$ at $(x_1 - 1, j)$ and $v_{|\mathcal{E}|,j}$ at $(x_{|\mathcal{E}|} + 1, j)$ for all $j = 0, \ldots, K - 1$.
- A source s and a sink t.

The positions of the vertices are only to help understanding. The edges are:

- $(v_{i,j}, v_{i+1,j})$ for all $i = 0, \ldots, |\mathcal{E}-1|$ and $j = 0, \ldots, K-1$, with length $w_e = 0$. If $y_{i+1} = j$, then the delay function $d_e = -r_{\alpha_{i+1}}$; otherwise, $d_e = 0$.
- $(v_{i,j}, v_{i,j'})$ for all $i = 1, \ldots, |\mathcal{E}-1|$ and $(j, j') \in \{0, \ldots, K-1\}^2$, with $w_e = 1$ and $d_e = 0$.
- $(s, v_{0,j})$ and $(v_{|\mathcal{E}|,j}, t)$ for all $j = 0, \ldots, K-1$, with $w_e = 1$ and $d_e = 0$.

The distance bound in the corresponding DncDa is the same as the maximum number of segments in GMG.

D Proof of Theorem 13

Theorem 13. *GMG is universally capability-positive if and only if both $r_g(\cdot)$ and $r_m(\cdot)$ are constant functions.*

Proof. If both r_g and r_m are constant functions, the same proof as in Theorem 9 applies to show that capability-positive holds universally here.

If r_g and r_m are not both constant functions, we show that there is always some instance of GMG with payoff r_g and r_m that is not capability-positive.

If r_g is not a constant function, let $v = \min\{x \mid r_g(x) \neq r_g(x+1)\}$. Denote $\rho = \frac{r_g(v+1)}{r_g(v)}$, then $\rho \neq 1$, and $r_g(v') = r_g(v)$ for all $v' \leq v$. We show that we can always construct a game with $n = v + 1$ players and $K = 2$ lines that is a counterexample for the capability-positive property with $b = 1$ (i.e. $W_1^+ > W_2^-$).

Case 1. $\rho < 1$ The layout of the constructed game is shown in Fig. 10. All the resources are gold. Block B is constructed as follows. Following the increasing order of x, $y_t = k$ means the t-th point is on line $y = k$. $N_0(t)$ ($N_1(t)$) denotes the number of points on line $y = 0$ ($y = 1$) within the first t points. Denote $D(t) = \rho N_0(t) - N_1(t)$. We use the following algorithm to position the gold:

1. `while` $N_0(t) < \frac{3}{1-\rho}$, do:
   If $D(t) \leq 1$, put $y_{t+1} = 0$; else, put $y_{t+1} = 1$;
   $t \leftarrow t + 1$;
2. `while` $D(t) \leq 1 + \rho$, do: $y_{t+1} = 0, t \leftarrow t + 1$.

Denote the total number of gold put down as N. Under the above construction, the following properties hold:

- For all $t = 1 \ldots N$, $D(t) > 0$ (all prefixes are better)
- For all $t = 0 \ldots N - 1$, $D(N) - D(t) > 0$ (all suffixes are better)
- $D(N) < 3$
- $N_0(N) > \frac{3}{1-\rho}$

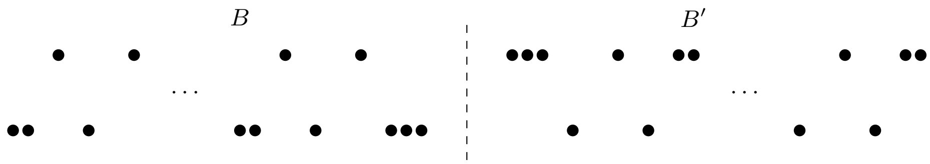


Fig. 10: Counterexample for the case $\rho < 1$ for r_g and $\rho > 1$ for r_m. Each dot is a resource. The upper (lower) line is $y = 1$ ($y = 0$). Block B and B' are centrosymmetric. The direction of increasing x is from left to right.

Block B' is obtained by flipping B in both x and y direction.

For $b = 2$, a PNE is all players choose $y = 0$ in B and $y = 1$ in B'. This is a PNE because of the prefix and suffix properties. Its social welfare $W_2 = (v+1) \cdot 2N_0(N)r_g(v+1)$. For $b = 1$, the PNEs are a players chooses $y = 0$ and $v+1-a$ players choose $y = 1$, where $1 \le a \le v$. All PNEs have the same social welfare $W_1 = (v+1) \cdot (N_0(N) + N_1(N))r_g(v)$. Then

$$\frac{W_1 - W_2}{(v+1)r_g(v)} = N_0(N) + N_1(N) - 2\rho N_0(N) = N_0(N)(1-\rho) - D(N) > 3 - 3 = 0.$$

Therefore $W_1 > W_2$, which implies $W_1^+ > W_2^-$.

Case 2. $\rho > 1$ The layout of the constructed game is in Fig. 11. We choose $N > \frac{\rho}{\rho-1}$. A PNE for $b = 2$ is all players choose the first N gold on $y = 0$ and the N gold on $y = 1$, which has welfare $W_2 = (v+1) \cdot 2Nr_g(v+1)$. A PNE for $b = 1$ is all players choose $y = 0$, which has welfare $W_1 = (v+1) \cdot (2N+1)r_g(v+1)$. Clearly $W_1 > W_2$, so $W_1^+ > W_2^-$.

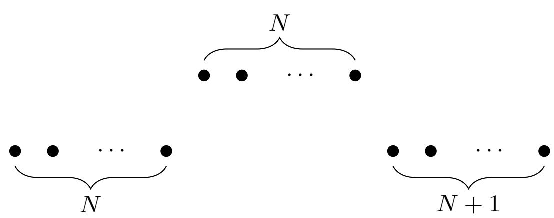


Fig. 11: Counterexample for the case $\rho > 1$ for r_g and $\rho < 1$ for r_m.

If r_m is not a constant function, let $v = \min\{i \mid r_m(i) \ne r_m(i+1)\}$. Denote $\rho = \frac{r_m(v+1)}{r_m(v)}$.

Case 1. $\rho < 1$ We use the same layout as in Fig. 11 with all resources being mines and $v+1$ players. We choose $N > \frac{\rho}{1-\rho}$. A PNE for $b = 2$ is all players chooses the bottom-right $N+1$ mines and avoiding all other mines, which has welfare $W_2 = (v+1)\cdot(N+1)r_m(v+1)$. A PNE for $b=1$ is all players choose $y = 1$, which has welfare $W_1 = (v+1)\cdot Nr_m(v+1)$. Clearly $W_1 > W_2$.

Case 2. $\rho > 1$ We use the same layout as in Fig. 10 with all resources being mines and $v+1$ players. Block B is constructed in a similar way with $D(t) = \frac{1}{\rho}N_0(t) - N_1(t)$, and the algorithm is:

1. `while` $N_1(t) < \frac{3}{1-1/\rho}$, do:
   If $D(t) \le 1$, put $y_{t+1} = 0$; else, put $y_{t+1} = 1$;
   $t \leftarrow t+1$;
2. `while` $D(t) \le 1 + \frac{1}{\rho}$, do: $y_{t+1} = 0, t \leftarrow t+1$.

The properties following the construction becomes:

– For all $t = 1 \dots N$, $D(t) > 0$ (all prefixes are better)
– For all $t = 0 \dots N-1$, $D(N) - D(t) > 0$ (all suffixes are better)
– $D(N) < 3$
– $N_1(N) > \frac{3\rho}{\rho-1}$

For $b = 2$, a PNE is all players choose $y = 1$ in B and $y = 0$ in B', whose social welfare $W_2 = (v+1)\cdot 2N_1(N)r_m(v+1)$. For $b = 1$, the PNEs are a players choose $y = 0$ and $v+1-a$ players choose $y = 1$, where $1 \le a \le v$. All PNEs have the same social welfare $W_1 = (v+1)\cdot(N_0(N) + N_1(N))r_m(v)$. Then

$$\frac{W_1 - W_2}{-(v+1)r_m(v)} = 2\rho N_1(N) - N_0(N) - N_1(N) = (\rho-1)N_1(N) - \rho D(N) > 3\rho - 3\rho = 0.$$

Therefore $W_1 > W_2$, which implies $W_1^+ > W_2^-$.

E Proofs for the alternating ordering game

E.1 Preliminaries

We first establish some notational convenience for the subsequent analyses:

– Following the increasing order of x, we use (x_t, y_t, α_t) to denote the location and type of the t-th resource.
– Each player's strategy can be represented as a function over the integer domain $[0, 4M+1]$, specifying the function value at each x_t for $t \in [0, 4M+1]$. We represent a function compactly as the set of intervals $\{[a_\xi^-, a_\xi^+]\}_{\xi=0}^{c-1}$ over t with $f(t) = 1$. For example, $f = \{[0,2],[5,5]\}$ represents the function of $f(t) = 1$ if $t \in [0,2]$ or $t = 5$, and $f(t) = 0$ otherwise.
– We use the **canonical** representation of f throughout the paper, where
  - a_ξ^-, a_ξ^+ are integers for all ξ, and $a_0^- \ge 0, a_{c-1}^+ \le 4M+1$;
  - $a_\xi^- \le a_\xi^+, a_{\xi+1}^- - a_\xi^+ \ge 2$ for all ξ, i.e. the representation uses the least number of intervals.

– Denote the set of functions with exactly k segments as $\mathcal{F}_k$. Then $\mathcal{L}_b = \bigcup_{k \leq b} \mathcal{F}_k$. The following general form covers all functions within $\mathcal{F}_k$:
  - For $k = 2c+1$ (odd k), $\mathcal{F}_k = \Big\{\{[a_\xi^-, a_\xi^+]\}_{\xi=0}^{c} \big| a_0^- = 0, a_c^+ = 4M + 1\Big\} \bigcup \Big\{\{[a_\xi^-, a_\xi^+]\}_{\xi=0}^{c-1} \big| a_0^- > 0, a_{c-1}^+ < 4M+1\Big\}$
  - For $k = 2c$ (even k), $\mathcal{F}_k = \Big\{\{[a_\xi^-, a_\xi^+]\}_{\xi=0}^{c-1} \big| a_0^- = 0, a_{c-1}^+ < 4M + 1\Big\} \bigcup \Big\{\{[a_\xi^-, a_\xi^+]\}_{\xi=0}^{c-1} \big| a_0^- > 0, a_{c-1}^+ = 4M+1\Big\}$

E.2 Proof of Theorem 16

Lemma 2. *Given an arbitrary set of strategies used by the other players $\boldsymbol{f}_{-i}$ and any $k_0 \geq 1$, denote $f_i^* = \arg\max_{f_i \in \mathcal{L}_{k_0}} u_i(f_i, \boldsymbol{f}_{-i})$ as the optimal strategy for player i, then $f_i^* = \{[a_\xi^-, a_\xi^+]\}_{\xi=0}^{c-1}$ must satisfy the following condition:*

$$\forall \xi, a_\xi^- = 0 \text{ or } 4j_\xi^- + 3 \text{ for some } j_\xi^-, a_\xi^+ = 4M+1 \text{ or } 4j_\xi^+ \text{ for some } j_\xi^+.$$

Proof. This lemma essentially states that the segments of an optimal strategy can only start and end at particular locations in the sequence. We prove this lemma by showing that if $f_i^* \in \mathcal{F}_k$ does not satisfy the above condition, then there exists f_i' which uses $k' \leq k$ segments and achieves a payoff $u_i' = u_i(f_i', \mathbf{f}_{-i})$ higher than $u_i^* = u_i(f_i^*, \mathbf{f}_{-i})$, therefore contradicting the fact that f_i^* is optimal.

If f_i^* does not satisfy the given condition, then either there exists a ξ_0 where $a_{\xi_0}^- \neq 0$ and $a_{\xi_0}^- \neq 4j+3$ for all j, or there exists a ξ_0 where $a_{\xi_0}^+ \neq 4M+1$ and $a_{\xi_0}^+ \neq 4j$ for all j. We consider each case separately here.

1. There exists a ξ_0 where $a_{\xi_0}^- \neq 0$ and $a_{\xi_0}^- \neq 4j+3$ for all j. Consider the value of $a_{\xi_0}^-$:
   – $a_{\xi_0}^- = 4j$ for some $j \neq 0$. Since f_i^* is in canonical form, $f_i^*(4j-1) = 0$. We know $y_{4j-1} = 0, \alpha_{4j-1} = \text{mine}$. Let f_i' be identical to f_i^* except changing $a_{\xi_0}^- = 4j-1$, then the payoff achieved by f_i' is $u_i' = u_i^* - \mu > u_i^*$. And the number of segments of f_i' $k' \leq k$.
   – $a_{\xi_0}^- = 4j+1$ for some j. We know $y_{4j+1} = 0, \alpha_{4j+1} = \text{gold}$. Let f_i' be identical to f_i^* except changing $a_{\xi_0}^- = 4j+2$ (if this makes interval ξ_0 empty, then remove interval ξ_0), then the payoff achieved by f_i' is $u_i' = u_i^* + r_g(x'_{4j+1}) > u_i^*$, and $k' \leq k$.
   – $a_{\xi_0}^- = 4j+2$ for some j. We know $y_{4j+2} = 1, \alpha_{4j+2} = \text{mine}$. Let f_i' be identical to f_i^* except changing $a_{\xi_0}^- = 4j+3$ (if this makes interval ξ_0 empty, then remove interval ξ_0), then the payoff achieved by f_i' is $u_i' = u_i^* - \mu > u_i^*$, and $k' \leq k$.
2. There exists a ξ_0 where $a_{\xi_0}^+ \neq 4M+1$ and $a_{\xi_0}^+ \neq 4j$ for all j. Consider the value of $a_{\xi_0}^+$:
   – $a_{\xi_0}^+ = 4j+1$ for some $j < M$. We know $y_{4j+1} = 0, \alpha_{4j+1} = \text{gold}$. Let f_i' be identical to f_i^* except changing $a_{\xi_0}^+ = 4j$, then the payoff achieved by f_i' is $u_i' = u_i^* + r_g(x'_{4j+1}) > u_i^*$, and $k' \leq k$.

- $a^+_{\xi_0} = 4j+2$ for some j. We know $y_{4j+2} = 1, \alpha_{4j+2} =$ mine. Let f'_i be identical to f^*_i except changing $a^+_{\xi_0} = 4j+1$, then the payoff achieved by f'_i is $u'_i = u^*_i - \mu > u^*_i$, and $k' \le k$.
- $a^+_{\xi_0} = 4j+3$ for some j. We know $y_{4j+4} = 1, \alpha_{4j+4} =$ gold. Let f'_i be identical to f^*_i except changing $a^+_{\xi_0} = 4j+4$, then the payoff achieved by f'_i is $u'_i = u^*_i + r_g(x'_{4j+4}) > u^*_i$, and $k' \le k$.

All the cases imply that f^*_i cannot be optimal within $\mathcal{L}_{k_0}$, which is a contradiction. Therefore f^*_i must satisfy the given condition.

Lemma 3. *Following Lemma 2, if the optimal strategy $f^*_i \in \mathcal{F}_k$ where $k \le 2M+1$, then it must has the following properties:*

- *If $k = 2c+1$ (odd k), then f^*_i satisfies either condition S_1 or condition S_2, and f^*_i always covers $M+c+1$ gold and $M-c$ mines.*
  - *S_1: $f^*_i = \{[a^-_\xi, a^+_\xi]\}^c_{\xi=0}$ where $a^-_0 = 0, a^+_c = 4M+1$, and $\{a^+_\xi = 4j^+_\xi\}^{c-1}_{\xi=0}$, $\{a^-_\xi = 4j^-_\xi + 3\}^c_{\xi=1}$, $\{j^-_\xi < j^+_\xi\}^{c-1}_{\xi=1}$, $\{j^+_\xi \le j^-_{\xi+1}\}^{c-1}_{\xi=0}$, $j^-_c < M$, $j^+_0 \ge 0$ for some $\{j^-_\xi\}^c_{\xi=1}$ and $\{j^+_\xi\}^{c-1}_{\xi=0}$.*
  - *S_2: $f^*_i = \{[a^-_\xi, a^+_\xi]\}^{c-1}_{\xi=0}$ where $\{a^-_\xi = 4j^-_\xi + 3\}^{c-1}_{\xi=0}$, $\{a^+_\xi = 4j^+_\xi\}^{c-1}_{\xi=0}$, $\{j^-_\xi < j^+_\xi\}^{c-1}_{\xi=0}$, $\{j^+_\xi \le j^-_{\xi+1}\}^{c-2}_{\xi=0}$, $j^+_{c-1} \le M$, $j^-_0 \ge 0$ for some $\{j^-_\xi\}^{c-1}_{\xi=0}$ and $\{j^+_\xi\}^{c-1}_{\xi=0}$.*
- *If $k = 2c$ (even k), then either f^*_i satisfies condition S_3, in which case f^*_i always covers $M+c+1$ gold and $M-c+1$ mines, or f^*_i satisfies condition S_4, in which case f^*_i always covers $M+c$ gold and $M-c$ mines.*
  - *S_3: $f^*_i = \{[a^-_\xi, a^+_\xi]\}^{c-1}_{\xi=0}$ where $a^-_0 = 0$, and $\{a^+_\xi = 4j^+_\xi\}^{c-1}_{\xi=0}$, $\{a^-_\xi = 4j^-_\xi + 3\}^{c-1}_{\xi=1}$, $\{j^-_\xi < j^+_\xi\}^{c-1}_{\xi=1}$, $\{j^+_\xi \le j^-_{\xi+1}\}^{c-2}_{\xi=0}$, $j^+_{c-1} \le M$, $j^+_0 \ge 0$ for some $\{j^-_\xi\}^{c-1}_{\xi=1}$ and $\{j^+_\xi\}^{c-1}_{\xi=0}$.*
  - *S_4: $f^*_i = \{[a^-_\xi, a^+_\xi]\}^{c-1}_{\xi=0}$ where $a^+_{c-1} = 4M+1$, and $\{a^+_\xi = 4j^+_\xi\}^{c-2}_{\xi=0}$, $\{a^-_\xi = 4j^-_\xi + 3\}^{c-1}_{\xi=0}$, $\{j^-_\xi < j^+_\xi\}^{c-2}_{\xi=0}$, $\{j^+_\xi \le j^-_{\xi+1}\}^{c-2}_{\xi=0}$, $j^-_0 \ge 0$, $j^-_{c-1} < M$ for some $\{j^-_\xi\}^{c-1}_{\xi=0}$ and $\{j^+_\xi\}^{c-2}_{\xi=0}$.*

Proof. We prove for each case.

1. $k = 2c+1$ (odd k)

Here, f^*_i is either of form $\{[a^-_\xi, a^+_\xi]\}^c_{\xi=0}$ where $a^-_0 = 0$ and $a^+_c = 4M+1$, or of form $\{[a^-_\xi, a^+_\xi]\}^{c-1}_{\xi=0}$ where $a^-_0 > 0$ and $a^+_{c-1} < 4M+1$.

1.1 If f^*_i is of form $\{[a^-_\xi, a^+_\xi]\}^c_{\xi=0}$ where $a^-_0 = 0$ and $a^+_c = 4M+1$, then according to Lemma 2, we have $\{a^+_\xi = 4j^+_\xi\}^{c-1}_{\xi=0}$ and $\{a^-_\xi = 4j^-_\xi + 3\}^c_{\xi=1}$ for some $\{j^-_\xi\}^c_{\xi=1}$ and $\{j^+_\xi\}^{c-1}_{\xi=0}$. And since f^*_i is canonical, we have $\{j^-_\xi < j^+_\xi\}^{c-1}_{\xi=1}$, $\{j^+_\xi \le j^-_{\xi+1}\}^{c-1}_{\xi=0}$, $j^-_c < M$, $j^+_0 \ge 0$. Therefore, in this case, f^*_i satisfies condition S_1.

Now consider the number of gold and mines covered by f^*_i that satisfies condition S_1. First consider $c \ge 1$. Segment $[a^-_\xi, a^+_\xi]$ (where f^*_i has value 1) covers $j^+_\xi - j^-_\xi$ gold and $j^+_\xi - j^-_\xi - 1$ mines, for $\xi = 1, \dots, c-1$. Segment

$[a^+_{\xi-1}+1, a^-_\xi - 1]$ (where f^*_i has value 0) covers $j^-_\xi - j^+_{\xi-1}+1$ gold and $j^-_\xi - j^+_{\xi-1}$ mines, for $\xi = 1, \ldots, c$. Segment $[a^-_0, a^+_0]$ covers $j^+_0 + 1$ gold and j^+_0 mines, and segment $[a^-_c, a^+_c]$ covers $M - j^-_c$ gold and $M - j^-_c - 1$ mines. Therefore, the number of gold covered by f^*_i is

$$j^+_0 + 1 + \sum_{\xi=1}^{c-1}(j^+_\xi - j^-_\xi) + \sum_{\xi=1}^{c}(j^-_\xi - j^+_{\xi-1} + 1) + M - j^-_c = M + c + 1,$$

and the number of mines covered by f^*_i is

$$j^+_0 + \sum_{\xi=1}^{c-1}(j^+_\xi - j^-_\xi - 1) + \sum_{\xi=1}^{c}(j^-_\xi - j^+_{\xi-1}) + M - j^-_c - 1 = M - c.$$

It's straightforward to check that the above expressions also hold for $c = 0$.

1.2 If f^*_i is of form $\{[a^-_\xi, a^+_\xi]\}_{\xi=0}^{c-1}$ where $a^-_0 > 0$ and $a^+_{c-1} < 4M+1$, then similar to case 1.1, Lemma 2 and the fact that f^*_i is canonical imply that f^*_i satisfies condition S_2.

Consider the number of gold and mines covered by f^*_i that satisfies condition S_2. Segment $[a^-_\xi, a^+_\xi]$ (where f^*_i has value 1) covers $j^+_\xi - j^-_\xi$ gold and $j^+_\xi - j^-_\xi - 1$ mines, for $\xi = 0, \ldots, c-1$. Segment $[a^+_{\xi-1}+1, a^-_\xi - 1]$ (where f^*_i has value 0) covers $j^-_\xi - j^+_{\xi-1} + 1$ gold and $j^-_\xi - j^+_{\xi-1}$ mines, for $\xi = 1, \ldots, c-1$. Segment $[0, a^-_0 - 1]$ (where f^*_i has value 0) covers $j^-_0 + 1$ gold and j^-_0 mines, and segment $[a^+_{c-1}+1, 4M+1]$ (where f^*_i has value 0) covers $M - j^+_{c-1} + 1$ gold and $M - j^+_{c-1}$ mines. Therefore, the number of gold covered by f^*_i is

$$j^-_0 + 1 + \sum_{\xi=0}^{c-1}(j^+_\xi - j^-_\xi) + \sum_{\xi=1}^{c-1}(j^-_\xi - j^+_{\xi-1} + 1) + M - j^+_{c-1} + 1 = M + c + 1,$$

and the number of mines covered by f^*_i is

$$j^-_0 + \sum_{\xi=0}^{c-1}(j^+_\xi - j^-_\xi - 1) + \sum_{\xi=1}^{c-1}(j^-_\xi - j^+_{\xi-1}) + M - j^+_{c-1} = M - c.$$

2. $k = 2c$ **(even** k**)**

Here, f^*_i is either of form $\{[a^-_\xi, a^+_\xi]\}_{\xi=0}^{c-1}$ where $a^-_0 = 0$ and $a^+_{c-1} < 4M+1$, or of form $\{[a^-_\xi, a^+_\xi]\}_{\xi=0}^{c-1}$ where $a^-_0 > 0$ and $a^+_{c-1} = 4M+1$.

2.1 If f^*_i is of form $\{[a^-_\xi, a^+_\xi]\}_{\xi=0}^{c-1}$ where $a^-_0 = 0$ and $a^+_{c-1} < 4M+1$, then similar to case 1.1, Lemma 2 and the fact that f^*_i is canonical imply that f^*_i satisfies condition S_3.

Consider the number of gold and mines covered by f^*_i that satisfies condition S_3. Segment $[a^-_\xi, a^+_\xi]$ (where f^*_i has value 1) covers $j^+_\xi - j^-_\xi$ gold and $j^+_\xi - j^-_\xi - 1$

mines, for $\xi = 1, \ldots, c-1$. Segment $[a^+_{\xi-1}+1, a^-_\xi - 1]$ (where f^*_i has value 0) covers $j^-_\xi - j^+_{\xi-1} + 1$ gold and $j^-_\xi - j^+_{\xi-1}$ mines, for $\xi = 1, \ldots, c-1$. Segment $[a^-_0, a^+_0]$ (where f^*_i has value 1) covers $j^+_0 + 1$ gold and j^+_0 mines, and segment $[a^+_{c-1}+1, 4M+1]$ (where f^*_i has value 0) covers $M - j^+_{c-1} + 1$ gold and $M - j^+_{c-1}$ mines. Therefore, the number of gold covered by f^*_i is

$$j^+_0 + 1 + \sum_{\xi=1}^{c-1}(j^+_\xi - j^-_\xi) + \sum_{\xi=1}^{c-1}(j^-_\xi - j^+_{\xi-1} + 1) + M - j^+_{c-1} + 1 = M + c + 1,$$

and the number of mines covered by f^*_i is

$$j^+_0 + \sum_{\xi=1}^{c-1}(j^+_\xi - j^-_\xi - 1) + \sum_{\xi=1}^{c-1}(j^-_\xi - j^+_{\xi-1}) + M - j^+_{c-1} = M - c + 1.$$

2.2 If f^*_i is of form $\{[a^-_\xi, a^+_\xi]\}_{\xi=0}^{c-1}$ where $a^-_0 > 0$ and $a^+_{c-1} = 4M+1$, then similar to case 1.1, Lemma 2 and the fact that f^*_i is canonical imply that f^*_i satisfies condition S_4.

Consider the number of gold and mines covered by f^*_i that satisfies condition S_4. Segment $[a^-_\xi, a^+_\xi]$ (where f^*_i has value 1) covers $j^+_\xi - j^-_\xi$ gold and $j^+_\xi - j^-_\xi - 1$ mines, for $\xi = 0, \ldots, c-2$. Segment $[a^+_{\xi-1}+1, a^-_\xi - 1]$ (where f^*_i has value 0) covers $j^-_\xi - j^+_{\xi-1} + 1$ gold and $j^-_\xi - j^+_{\xi-1}$ mines, for $\xi = 1, \ldots, c-1$. Segment $[0, a^-_0 - 1]$ (where f^*_i has value 0) covers $j^-_0 + 1$ gold and j^-_0 mines, and segment $[a^-_{c-1}, a^+_{c-1}]$ (where f^*_i has value 1) covers $M - j^-_{c-1}$ gold and $M - j^-_{c-1} - 1$ mines. Therefore, the number of gold covered by f^*_i is

$$j^-_0 + 1 + \sum_{\xi=0}^{c-2}(j^+_\xi - j^-_\xi) + \sum_{\xi=1}^{c-1}(j^-_\xi - j^+_{\xi-1} + 1) + M - j^-_{c-1} = M + c,$$

and the number of mines covered by f^*_i is

$$j^-_0 + \sum_{\xi=0}^{c-2}(j^+_\xi - j^-_\xi - 1) + \sum_{\xi=1}^{c-1}(j^-_\xi - j^+_{\xi-1}) + M - j^-_{c-1} - 1 = M - c.$$

Lemma 4. *If one player (call it player A) covers g_A gold, player B covers g_B gold and m_B mines, and $g_A + g_B \geq 2M + 2$, then there is an upper bound on player B's payoff:*

$$u_B \leq (1-\rho)(2M + 2 - g_A) + \rho g_B + \mu m_B$$

Proof. Among the gold covered by player B, denote the number of them also covered by player A as d. Since the total number of gold is $2M + 2$, we have $g_A + g_B - d \leq 2M + 2$, i.e. $d \geq g_A + g_B - 2M - 2$. Therefore,

$$\begin{aligned} s_B &= g_B - d + d\rho + m_B\mu \\ &\leq g_B - (1-\rho)(g_A + g_B - 2M - 2) + m_B\mu \\ &= (1-\rho)(2M + 2 - g_A) + \rho g_B + \mu m_B. \end{aligned}$$

We define that a strategy achieves *best coverage* if it covers all the gold that is not covered by the other player.

Lemma 5. *Given one player (call it player A) covers g_A gold, if a strategy f_B for player B covers g_B gold and m_B mines and achieves best coverage, then any strategy f'_B that covers g'_B gold and m'_B mines will achieve a lower payoff than f_B, if*

$$g'_B \leq g_B \wedge m'_B > m_B, \text{ or } g'_B < g_B \wedge m'_B \geq m_B$$

Proof. Since f_B achieves best coverage, it covers $2M+2-g_A$ gold that is not covered by player A, and $g_B + g_A - 2M - 2$ gold that is covered by A. So the payoff achieved by f_B is

$$\begin{aligned} u_B &= 2M + 2 - g_A + \rho(g_B + g_A - 2M - 2) + m_B\mu \\ &= (1-\rho)(2M+2-g_A) + \rho g_B + \mu m_B. \end{aligned}$$

Consider the payoff of f'_B. By Lemma 4,

$$u'_B \leq (1-\rho)(2M+2-g_A) + \rho g'_B + \mu m'_B.$$

Since $\rho > 0$ and $\mu < 0$, we can see that if $g'_B \leq g_B \wedge m'_B > m_B$, or $g'_B < g_B \wedge m'_B \geq m_B$,

$$u'_B < (1-\rho)(2M+2-g_A) + \rho g_B + \mu m_B = u_B.$$

Lemma 6. *If both players' strategy space is $\mathcal{L}_b$ ($b \leq 2M+2$), then for all PNE (f_1^*, f_2^*), $f_1^*, f_2^* \in \mathcal{F}_b$, i.e. both strategies in the equilibria must use exactly b segments.*

Proof. We prove by induction.

Base case For $b = 1$, there is only two possible strategies in $\mathcal{L}_1$: $f^0 = \{\}$ and $f^1 = \{[0, 4M+1]\}$. Both uses exactly 1 segment. So the statement holds.

Induction step Consider the case $b = k$. First we show that if one of the strategies in a PNE uses exactly k segments, then the other strategy must also use exactly k segments. Without loss of generality, let $f_1^* \in \mathcal{F}_k$.

1. $k = 2c+1$ (odd k)

By Lemma 3, f_1^* must satisfy condition S_1 or S_2.

1.1 If f_1^* satisfies condition S_1, let $f_1^* = \{[a_\xi^-, a_\xi^+]\}_{\xi=0}^{c}$ where $a_0^- = 0, a_c^+ = 4M+1$, $\{a_\xi^+ = 4j_\xi^+\}_{\xi=0}^{c-1}$, $\{a_\xi^- = 4j_\xi^- + 3\}_{\xi=1}^{c}$, and f_1^* covers $M+c+1$ gold and $M-c$ mines. We construct $\hat{f}_2 = \{[\hat{a}_\xi^-, \hat{a}_\xi^+]\}_{\xi=0}^{c-1}$ according to f_1^* by setting $\{\hat{a}_\xi^- = 4j_\xi^+ + 3, \hat{a}_\xi^+ = 4\cdot\max(j_{\xi+1}^-, j_\xi^+ + 1)\}_{\xi=0}^{c-1}$ (note that here j_ξ^+ and j_ξ^- are the values used by f_1^*). It's easy to check that $\hat{f}_2$ satisfies condition S_2 and covers

$g_2 = M + c + 1$ gold and $m_2 = M - c$ mines. In particular, among the gold covered by $\hat{f}_2$, $2c$ of them are also covered by f_1^*, and $M - c + 1$ of them are covered by $\hat{f}_2$ only. Therefore, $\hat{f}_2$ achieves best coverage. $\hat{f}_2$ achieves a payoff of

$$\hat{u}_2 = M - c + 1 + 2c\rho + (M - c)\mu.$$

We show here that any $f_2' \in \mathcal{F}_{k'}$ where $k' < k$ will achieve a payoff $u_2' < \hat{u}_2$, therefore f_2^* must use exactly k segments. If $k' = 2c' + 1$, then $c' \leq c - 1$, and by Lemma 3, f_2' covers $g_2' = M + c' + 1$ gold and $m_2' = M - c'$ mines. We have $g_2' < g_2$ and $m_2' > m_2$. Therefore by Lemma 5, $u_2' < \hat{u}_2$.

If $k' = 2c'$, then $c' \leq c$. By Lemma 3, f_2' either covers $g_2' = M + c' + 1$ gold and $m_2' = M - c' + 1$ mines, in which case $g_2' \leq g_2$ and $m_2' > m_2$, or $g_2' = M + c'$ gold and $m_2' = M - c'$ mines, in which case $g_2' < g_2$ and $m_2' \geq m_2$. Therefore by Lemma 5, $u_2' < \hat{u}_2$.

1.2 By symmetry, the above proof also applies to the case where f_1^* satisfies condition S_2 (symmetry with respect to inverting the direction of x and y axis).

2. $k = 2c$ (even k), $c \leq M$

By Lemma 3, f_1^* must satisfy condition S_3 or S_4.

2.1 If f_1^* satisfies condition S_3, let $f_1^* = \{[a_\xi^-, a_\xi^+]\}_{\xi=0}^{c-1}$ where $a_0^- = 0$, and $\{a_\xi^+ = 4j_\xi^+\}_{\xi=0}^{c-1}$, $\{a_\xi^- = 4j_\xi^- + 3\}_{\xi=1}^{c-1}$, and f_1^* covers $M+c+1$ gold and $M-c+1$ mines. We construct $\hat{f}_2 = \{[\hat{a}_\xi^-, \hat{a}_\xi^+]\}_{\xi=0}^{c-1}$ according to f_1^* by sequentially setting $\hat{a}_0^-, \hat{a}_0^+, \hat{a}_1^-, \hat{a}_1^+, \ldots, \hat{a}_{c-1}^-, \hat{a}_{c-1}^+$ with $\hat{a}_0^- = \max(4j_0^+ - 1, 3)$, $\{\hat{a}_\xi^- = \max(4j_\xi^+ - 1, \hat{a}_{\xi-1}^+ + 3)\}_{\xi=1}^{c-1}$, $\{\hat{a}_\xi^+ = \max(4 \cdot j_{\xi+1}^-, \hat{a}_\xi^- + 1)\}_{\xi=0}^{c-2}$, $\hat{a}_{c-1}^+ = 4M + 1$. This $\hat{f}_2$ satisfies condition S_4 and covers $g_2 = M + c$ gold and $m_2 = M - c$ mines. In particular, among the gold covered by $\hat{f}_2$, $2c - 1$ of them are also covered by f_1^*, and $M - c + 1$ of them are covered by $\hat{f}_2$ only. Therefore, $\hat{f}_2$ achieves best coverage. $\hat{f}_2$ achieves a payoff of

$$\hat{u}_2 = M - c + 1 + (2c - 1)\rho + (M - c)\mu.$$

We show here that any $f_2' \in \mathcal{F}_{k'}$ where $k' < k$ will achieve a payoff $u_2' < \hat{u}_2$, therefore f_2^* must use exactly k segments. If $k' = 2c' + 1$, then $c' \leq c - 1$, and by Lemma 3, f_2' covers $g_2' = M + c' + 1$ gold and $m_2' = M - c'$ mines. We have $g_2' \leq g_2$ and $m_2' > m_2$. Therefore by Lemma 5, $u_2' < \hat{u}_2$.

If $k' = 2c'$, then $c' \leq c - 1$. By Lemma 3, f_2' either covers $g_2' = M + c' + 1$ gold and $m_2' = M - c' + 1$ mines, in which case $g_2' \leq g_2$ and $m_2' > m_2$, or $g_2' = M + c'$ gold and $m_2' = M - c'$ mines, in which case $g_2' < g_2$ and $m_2' > m_2$. Therefore by Lemma 5, $u_2' < \hat{u}_2$.

2.2 If f_1^* satisfies condition S_4, let $f_1^* = \{[a_\xi^-, a_\xi^+]\}_{\xi=0}^{c-1}$ where $a_{c-1}^+ = 4M + 1$, and $\{a_\xi^+ = 4j_\xi^+\}_{\xi=0}^{c-2}$, $\{a_\xi^- = 4j_\xi^- + 3\}_{\xi=0}^{c-1}$, and f_1^* covers $M + c$ gold and $M - c$ mines. We construct $\hat{f}_2 = \{[\hat{a}_\xi^-, \hat{a}_\xi^+]\}_{\xi=0}^{c-1}$ according to f_1^* by setting $\hat{a}_0^- = 0, \hat{a}_0^+ =$

$4j_0^-, \{\hat{a}_\xi^- = 4j_{\xi-1}^+ + 3, \hat{a}_\xi^+ = 4 \cdot \max(j_\xi^-, j_{\xi-1}^+ + 1)\}_{\xi=1}^{c-1}$. This $\hat{f}_2$ satisfies condition S_3 and covers $g_2 = M + c + 1$ gold and $m_2 = M - c + 1$ mines. In particular, among the gold covered by $\hat{f}_2$, $2c-1$ of them are also covered by f_1^*, and $M-c+2$ of them are covered by $\hat{f}_2$ only. Therefore, $\hat{f}_2$ achieves best coverage. $\hat{f}_2$ achieves a payoff of

$$\hat{u}_2 = M - c + 2 + (2c-1)\rho + (M - c + 1)\mu.$$

We show here that any $f_2' \in \mathcal{F}_{k'}$ where $k' < k$ will achieve a payoff $u_2' < \hat{u}_2$, therefore f_2^* must use exactly k segments. If $k' = 2c' + 1$, then $c' \le c - 1$, and by Lemma 3, f_2' covers $g_2' = M + c' + 1$ gold and $m_2' = M - c'$ mines. We have $g_2' < g_2$ and $m_2' \ge m_2$. Therefore by Lemma 5, $u_2' < \hat{u}_2$.

If $k' = 2c'$, then $c' \le c-1$. By Lemma 3, f_2' either covers $g_2' = M + c' + 1$ gold and $m_2' = M - c' + 1$ mines, in which case $g_2' < g_2$ and $m_2' > m_2$, or $g_2' = M + c'$ gold and $m_2' = M - c'$ mines, in which case $g_2' < g_2$ and $m_2' \ge m_2$. Therefore by Lemma 5, $u_2' < \hat{u}_2$.

3. $k = 2c$ **(even** k**),** $c = M + 1$

By Lemma 3, f_1^* must satisfy condition S_3 or S_4. In fact, in this case, no function satisfies S_4, and there is only one function satisfies S_3, which is the function that covers all $2M + 2$ gold and no mine. Construct $\hat{f}_2$ to be the same as f_1^*, which covers all $g_2 = 2M + 2$ gold and $m_2 = 0$ mine. Since f_1^* already covers all gold, $\hat{f}_2$ trivially achieves best coverage.

We show here that any $f_2' \in \mathcal{F}_{k'}$ where $k' < k$ will achieve a payoff $u_2' < \hat{u}_2$, therefore f_2^* must use exactly k segments. If $k' = 2c' + 1$, then $c' \le c - 1$, and by Lemma 3, f_2' covers $g_2' = M + c' + 1$ gold and $m_2' = M - c'$ mines. We have $g_2' < g_2$ and $m_2' \ge m_2$. Therefore by Lemma 5, $u_2' < \hat{u}_2$.

If $k' = 2c'$, then $c' \le c-1$. By Lemma 3, f_2' either covers $g_2' = M + c' + 1$ gold and $m_2' = M - c' + 1$ mines, in which case $g_2' < g_2$ and $m_2' > m_2$, or $g_2' = M + c'$ gold and $m_2' = M - c'$ mines, in which case $g_2' < g_2$ and $m_2' \ge m_2$. Therefore by Lemma 5, $u_2' < \hat{u}_2$.

Now we have shown that if one of the strategies in a PNE uses exactly k segments, then the other strategy must also use exactly k segments. What is left to show is that there is no PNE where both strategies use less than k segments.

We prove by contradiction. Assume there is a PNE (f_1^*, f_2^*) where both f_1^* and f_2^* use less than k segments. By the induction hypothesis, $f_1^*, f_2^* \in \mathcal{F}_{k-1}$.

1. $k - 1 = 2c + 1$ **(even** k**),** $c \le M$

By Lemma 3, f_1^* must satisfy condition S_1 or S_2. If f_1^* satisfies condition S_1, let $f_1^* = \{[a_\xi^-, a_\xi^+]\}_{\xi=0}^{c}$ where $a_0^- = 0, a_c^+ = 4M + 1$, $\{a_\xi^+ = 4j_\xi^+\}_{\xi=0}^{c-1}$, $\{a_\xi^- = 4j_\xi^- + 3\}_{\xi=1}^{c}$. We construct $\hat{f}_2 = \{[\hat{a}_\xi^-, \hat{a}_\xi^+]\}_{\xi=0}^{c}$ according to f_1^* by setting $\hat{a}_0^- = 0, \hat{a}_0^+ = 0, \{\hat{a}_\xi^- = 4j_{\xi-1}^+ + 3, \hat{a}_\xi^+ = 4 \cdot \max(j_\xi^-, j_{\xi-1}^+ + 1)\}_{\xi=1}^{c}$. It is easy to check that $\hat{f}_2$ uses k segments, covers $g_2 = M + c + 2$ gold and $m_2 = M - c$ mines, and achieves best coverage. By Lemma 3, f_2^* covers $g_2^* = M + c + 1$ gold and

$m_2^* = M - c$ mines. Thus $g_2^* < g_2$ and $m_2^* \geq m_2$. By Lemma 5, $u_2^* < \hat{u}_2$, therefore (f_1^*, f_2^*) cannot be a PNE, contradiction.

By symmetry, the above proof also applies to the case where f_1^* satisfies condition S_2 (symmetry with respect to inverting the direction of x and y axis).

2. $k - 1 = 2c$ **(odd** k**),** $c \leq M$

By Lemma 3, f_1^* must satisfy condition S_3 or S_4. If f_1^* satisfies condition S_3, let $f_1^* = \{[a_\xi^-, a_\xi^+]\}_{\xi=0}^{c-1}$ where $a_0^- = 0$, and $\{a_\xi^+ = 4j_\xi^+\}_{\xi=0}^{c-1}$, $\{a_\xi^- = 4j_\xi^- + 3\}_{\xi=1}^{c-1}$. We construct $\hat{f}_2 = \{[\hat{a}_\xi^-, \hat{a}_\xi^+]\}_{\xi=0}^{c}$ according to f_1^* by sequentially setting $\hat{a}_0^-, \hat{a}_0^+, \hat{a}_1^-, \hat{a}_1^+, \ldots, \hat{a}_c^-, \hat{a}_c^+$ with $\hat{a}_0^- = 0$, $\hat{a}_0^+ = 0$, $\hat{a}_1^- = \max(4j_0^+ - 1, 3)$, $\{\hat{a}_\xi^- = \max(4j_{\xi-1}^+ - 1, \hat{a}_{\xi-1}^+ + 3)\}_{\xi=2}^{c}$, $\{\hat{a}_\xi^+ = \max(4 \cdot j_\xi^-, \hat{a}_\xi^- + 1)\}_{\xi=1}^{c-1}$, $\hat{a}_c^+ = 4M + 1$. It is easy to check that $\hat{f}_2$ uses k segments, covers $g_2 = M + c + 1$ gold and $m_2 = M - c$ mines, and achieves best coverage. By Lemma 3, f_2^* either covers $g_2^* = M + c + 1$ gold and $m_2^* = M - c + 1$ mines, in which case $g_2^* \leq g_2$ and $m_2^* > m_2$, or $g_2^* = M + c$ gold and $m_2^* = M - c$ mines, in which case $g_2^* < g_2$ and $m_2^* \geq m_2$. Therefore by Lemma 5, $u_2^* < \hat{u}_2$, which means (f_1^*, f_2^*) cannot be a PNE, contradiction.

If f_1^* satisfies condition S_4, let $f_1^* = \{[a_\xi^-, a_\xi^+]\}_{\xi=0}^{c-1}$ where $a_{c-1}^+ = 4M + 1$, and $\{a_\xi^+ = 4j_\xi^+\}_{\xi=0}^{c-2}$, $\{a_\xi^- = 4j_\xi^- + 3\}_{\xi=0}^{c-1}$. f_1^* covers $g_1^* = M + c$ gold and $m_1^* = M - c$ mines. We construct $\hat{f}_1 = \{[\hat{a}_\xi^-, \hat{a}_\xi^+]\}_{\xi=0}^{c}$ according to f_1^* by setting $\hat{a}_0^- = 0, \hat{a}_0^+ = 0, \{\hat{a}_\xi^- = a_{\xi-1}^-, \hat{a}_\xi^+ = a_{\xi-1}^+\}_{\xi=1}^{c}$, i.e. $\hat{f}_1$ is identical to f_1^* except $\hat{f}_1(0) = 1$. $\hat{f}_1$ uses k segments, and covers exactly the same set of gold and mines as f_1^* plus the gold at $t = 0$. Therefore, $\hat{f}_1$'s payoff is strictly higher than f_1^*'s payoff. This means (f_1^*, f_2^*) cannot be a PNE, contradiction.

This finishes the proof that there exists no PNE where both strategies use less than k segments. We have also shown that if one strategy in a PNE uses k segments, the other strategy must also use k segments. This together shows that for all PNE, both strategies in the equilibria must use exactly k segments. This finishes the proof by induction.

Theorem 16. *If* $-2 + \rho < \mu < -\rho$*, then for any level* b *strategy space* $\mathcal{L}_b$*, all PNEs have the same social welfare*

$$W_{\text{Equil}}(b) = \begin{cases} (2M+1)(1+\mu) + 2(1-\rho) + (2\rho - \mu - 1)b & \text{if } b \leq 2M+1 \\ (4M+4)\rho & \text{if } b \geq 2M+2 \end{cases}.$$

Proof. We consider different values of b.

1. $b = 2c + 1, 0 \leq c \leq M$

By Lemma 6, both f_1^* and f_2^* use exactly b segments, i.e. $f_1^*, f_2^* \in \mathcal{F}_b$. By Lemma 3, both f_1^* and f_2^* must satisfy condition S_1 or S_2, and each covers $M + c + 1$ gold and $M - c$ mines. If f_1^* satisfies S_1, denote $f_1^* = \{[a_\xi^-, a_\xi^+]\}_{\xi=0}^{c}$ where $a_0^- = 0, a_c^+ = 4M + 1$, $\{a_\xi^+ = 4j_\xi^+\}_{\xi=0}^{c-1}$, $\{a_\xi^- = 4j_\xi^- + 3\}_{\xi=1}^{c}$. Same as in the proof of Lemma 6, we construct $\hat{f}_2 = \{[\hat{a}^-_\xi, \hat{a}^+_\xi]\}_{\xi=0}^{c-1}$ according to f_1^* by setting $\{\hat{a}^-_\xi = 4j^+_\xi + 3, \hat{a}^+_\xi = 4 \cdot \max(j^-_{\xi+1}, j^+_\xi + 1)\}_{\xi=0}^{c-1}$. $\hat{f}_2 \in \mathcal{F}_b$ achieves best coverage and a payoff of $\hat{u}_2 = M-c+1+2c\rho+(M-c)\mu$. Since f_2^* always covers $M+c+1$ gold and $M-c$ mines, by Lemma 4, f_2^*'s payoff $u_2^* \leq \hat{u}_2$. But by definition of Nash equilibrium, $u_2^* \geq \hat{u}_2$. Therefore, $u_2^* = \hat{u}_2$, i.e. all f_2^* must achieve the same payoff of $M-c+1+2c\rho+(M-c)\mu$.

By symmetry (with respect to inverting the direction of x and y axis), the above proof can also be applied to show that if f_1^* satisfies S_2, then all f_2^* must achieve the same payoff of $M-c+1+2c\rho+(M-c)\mu$.

Therefore, in all cases, $u_2^* = M-c+1+2c\rho+(M-c)\mu$. Similarly, $u_1^* = M-c+1+2c\rho+(M-c)\mu$. So

$$\begin{aligned} W_{\text{Equil}}(b) &= 2M(1+\mu)+2+2(2\rho-\mu-1)c \\ &= 2M(1+\mu)+2+(2\rho-\mu-1)(b-1) \\ &= (2M+1)(1+\mu)+2(1-\rho)+(2\rho-\mu-1)b. \end{aligned}$$

2. $b = 2c, 1 \leq c \leq M$

By Lemma 6, $f_1^*, f_2^* \in \mathcal{F}_b$. By Lemma 3, both f_1^* and f_2^* must satisfy condition S_3 or S_4. If f_1^* satisfies S_3, denote $f_1^* = \{[a^-_\xi, a^+_\xi]\}_{\xi=0}^{c-1}$ where $a^-_0 = 0$, and $\{a^+_\xi = 4j^+_\xi\}_{\xi=0}^{c-1}$, $\{a^-_\xi = 4j^-_\xi + 3\}_{\xi=1}^{c-1}$, and f_1^* covers $M+c+1$ gold and $M-c+1$ mines. Same as in the proof of Lemma 6, we construct $\hat{f}_2 = \{[\hat{a}^-_\xi, \hat{a}^+_\xi]\}_{\xi=0}^{c-1}$ according to f_1^* by sequentially setting $\hat{a}^-_0, \hat{a}^+_0, \hat{a}^-_1, \hat{a}^+_1, \ldots, \hat{a}^-_{c-1}, \hat{a}^+_{c-1}$ with $\hat{a}^-_0 = \max(4j^+_0 - 1, 3)$, $\{\hat{a}^-_\xi = \max(4j^+_\xi - 1, \hat{a}^+_{\xi-1} + 3)\}_{\xi=1}^{c-1}$, $\{\hat{a}^+_\xi = \max(4 \cdot j^-_{\xi+1}, \hat{a}^-_\xi + 1)\}_{\xi=0}^{c-2}$, $\hat{a}^+_{c-1} = 4M+1$. This $\hat{f}_2$ satisfies condition S_4 and achieves a payoff of $\hat{u}_2 = M-c+1+(2c-1)\rho+(M-c)\mu$. For any f'_2 that satisfies S_3, by Lemma 3, it covers $M+c+1$ gold and $M-c+1$ mines. By Lemma 4, such f'_2's payoff

$$\begin{aligned} u'_2 &\leq (1-\rho)(M-c+1)+\rho(M+c+1)+\mu(M-c+1) \\ &= M-c+1+2c\rho+(M-c+1)\mu \\ &= \hat{u}_2+\rho+\mu < \hat{u}_2. \end{aligned}$$

By the definition of Nash equilibrium, $u_2^* \geq \hat{u}_2$, so f_2^* cannot satisfy S_3. Therefore, f_2^* must satisfy S_4, and by Lemma 3, f_2^* covers $M+c$ gold and $M-c$ mines. So by Lemma 4, $u_2^* \leq \hat{u}_2$. Therefore, $u_2^* = \hat{u}_2$, i.e. f_2^* always satisfies S_4 and achieves a payoff of $u_2^* = M-c+1+(2c-1)\rho+(M-c)\mu$.

If f_1^* satisfies S_4, denote $f_1^* = \{[a^-_\xi, a^+_\xi]\}_{\xi=0}^{c-1}$ where $a^+_{c-1} = 4M+1$, and $\{a^+_\xi = 4j^+_\xi\}_{\xi=0}^{c-2}$, $\{a^-_\xi = 4j^-_\xi + 3\}_{\xi=0}^{c-1}$, and f_1^* covers $M+c$ gold and $M-c$ mines. We construct $\hat{f}_2 = \{[\hat{a}^-_\xi, \hat{a}^+_\xi]\}_{\xi=0}^{c-1}$ according to f_1^* by setting $\hat{a}^-_0 = 0, \hat{a}^+_0 = 4j^-_0, \{\hat{a}^-_\xi = 4j^+_{\xi-1} + 3, \hat{a}^+_\xi = 4 \cdot \max(j^-_\xi, j^+_{\xi-1} + 1)\}_{\xi=1}^{c-1}$. This $\hat{f}_2$ satisfies condition S_3 and achieves a payoff of $\hat{u}_2 = M-c+2+(2c-1)\rho+(M-c+1)\mu$. For any f'_2 that satisfies S_4, by Lemma 3, it covers $M+c$ gold and $M-c$ mines. Denote d as the number of gold that is covered by both f_1^* and f'_2, noting that both f_1^* and f'_2 cannot cover the gold at $t=0$ and $t=4M+1$, we have $M+c+M+c-d \leq 2M$, so $d \geq 2c$. Therefore, such f_2''s payoff

$$\begin{aligned} u_2' &= M + c - d + d\rho + (M - c)\mu \\ &\leq M + c - 2c(1 - \rho) + (M - c)\mu \\ &= \hat{u}_2 - 2 + \rho - \mu < \hat{u}_2. \end{aligned}$$

By the definition of Nash equilibrium, $u_2^* \geq \hat{u}_2$, so f_2^* cannot satisfy S_4. Therefore, f_2^* must satisfy S_3, and by Lemma 3, f_2^* covers $M+c+1$ gold and $M-c+1$ mines. So by Lemma 4, $u_2^* \leq \hat{u}_2$. Therefore, $u_2^* = \hat{u}_2$, i.e. f_2^* always satisfies S_3 and achieves a payoff of $u_2^* = M - c + 2 + (2c - 1)\rho + (M - c + 1)\mu$.

Combining the above results, we can show that for any PNE (f_1^*, f_2^*), one of f_1^* and f_2^* must satisfy S_3 and achieves a payoff of $M - c + 2 + (2c - 1)\rho + (M - c + 1)\mu$, and the other must satisfy S_4 and achieves a payoff of $M - c + 1 + (2c - 1)\rho + (M - c)\mu$. Therefore,

$$\begin{aligned} W_{\text{Equil}}(b) &= (2M + 1)(1 + \mu) + 2(1 - \rho) + 2(2\rho - \mu - 1)c \\ &= (2M + 1)(1 + \mu) + 2(1 - \rho) + (2\rho - \mu - 1)b. \end{aligned}$$

3. $b \geq 2M + 2$

Since f_1^* and f_2^* can have at most $2M + 2$ segments, when $b \geq 2M + 2$, $f_1^*, f_2^* \in \mathcal{F}_{2M+2}$. There is only one function in $\mathcal{F}_{2M+2}$, which is the function that covers all gold and no mines, therefore both f_1^* and f_2^* must be this particular function. So $u_1^* = u_2^* = (2M + 2)\rho$, $W_{\text{Equil}}(b) = (4M + 4)\rho$.

E.3 Best social welfare by centralized solutions

Proposition 1. *For the alternating ordering game, the best social welfare achieved by any centralized solution under $\mathcal{L}_b$ is*

$$W_{\text{best}}(b) = \begin{cases} 2M + 2 + (2M + 1)\mu - \mu b & \text{if } b \leq 2M + 1 \\ 2M + 2 & \text{if } b \geq 2M + 2 \end{cases}.$$

Proof. First, we notice that a function with c changes from $y = 0$ to $y = 1$ covers at least $M - c$ mines.

For $b = 2c, c \leq M$, we can construct two functions $(\hat{f}_1, \hat{f}_2)$ achieving $W_{\text{best}}(b)$. $\hat{f}_1$ starts from $y = 1$ and ends at $y = 0$, covers all gold on $y = 1$ except the rightmost one, and the rightmost gold on $y = 0$; $\hat{f}_2$ starts from $y = 0$ and ends at $y = 1$ covers all gold on $y = 0$ except the rightmost one, and the rightmost gold on $y = 1$. Both use the line changes to avoid mines, with $\hat{f}_1$ covering $M - c + 1$ mines and $\hat{f}_2$ covering $M - c$ mines. No other functions can achieve a better welfare, since they cannot jointly cover more gold, and $w_g = x r_g(x)$ attains its maximum at $x = 1$. And to jointly cover fewer mines, both functions need to start from $y = 0$ and ends at $y = 1$, which can reduce the number of mines covered by at most 1. But then the two functions can only jointly cover at most $2M$ gold, which makes the social welfare lower than that of $(\hat{f}_1, \hat{f}_2)$.

For $b = 2c + 1, c \leq M$, we can construct two functions $(\hat{f}_1, \hat{f}_2)$ achieving $W_{\text{best}}(b)$. They jointly cover all gold with no overlap, and $M - c$ mines each. The construction is simply let $\hat{f}_1$ covers all gold on $y = 1$, $\hat{f}_2$ covers all gold on $y = 0$. No other functions can achieve a better welfare, since they cannot jointly cover more gold or less mines.

For $b \geq 2M + 2$, let $\hat{f}_1$ covers all gold on $y = 1$ and no mines, and $\hat{f}_2$ covers all gold on $y = 0$ and no mines. This achieves $W_{best}(b)$, which is in fact the maximum possible social welfare of this game.

E.4 Necessary condition for all PNEs having the same social welfare for any b and M

Proposition 2. *$-2 + \rho < \mu < -\rho$ is also a necessary condition for all PNEs having the same social welfare for any b and M.*

Proof. We provide constructions showing that if this condition is not satisfied, there is always some b and M where different PNEs have different social welfare. If $\mu \geq -\rho$, for $M > \frac{2(\rho+\mu)}{1-\rho} + 1$ and $b = 2$, $(\{[0, 0]\}, \{[0, 4M]\})$ is a PNE, $(\{[0, 4 \cdot \lfloor M/2 \rfloor]\}, \{[4 \cdot \lfloor M/2 \rfloor + 3, 4M + 1]\})$ is another PNE, and their social welfare is different. If $\mu \leq -2 + \rho$, for $M > \frac{2(-\mu-\rho)}{1-\rho}$ and $b = 2$, $(\{[4M - 1, 4M + 1]\}, \{[3, 4M + 1]\})$ is a PNE, $(\{[4 \cdot \lfloor M/2 \rfloor + 3, 4M + 1]\}, \{[0, 4 \cdot \lfloor M/2 \rfloor]\})$ is a PNE, and their social welfare is different.